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Random variables: $X(\omega)$.
Distribution: $Pr[X = x]$
$\sum_{x} Pr[X = x] = 1$.

Random Variable: $X$
Event: $A = [a, b]$, $Pr[X \in A]$,
CDF: $F(x) = Pr[X \leq x]$.
PDF: $f(x) = \frac{dF(x)}{dx}$.
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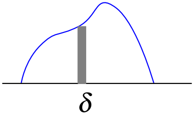
Continuous as Discrete.
$Pr[X \in [x, x+\delta]] \approx f(x)\delta$



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Product Rule: $Pr[A \cap B] = Pr[A|B]Pr[B]$.
Bayes Rule: $Pr[A|B] = Pr[B|A]Pr[A]/Pr[B]$.

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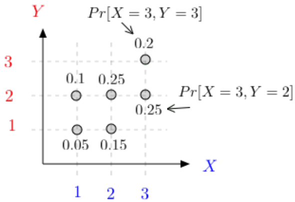
All work for continuous with intervals as events.

# Joint distribution.

| Y/X | 1 | 2 | 4 | 8 | |
|---|---|---|---|---|---|
| 1 | .03 | .05 | .1 | .02 | .20 |
| 2 | .2 | .01 | .03 | .02 | .26 |
| 3 | .21 | .06 | .03 | .01 | .31 |
| 5 | .02 | .2 | .02 | .01 | .25 |
| | .44 | .32 | .18 | .06 | |



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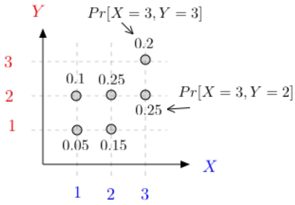

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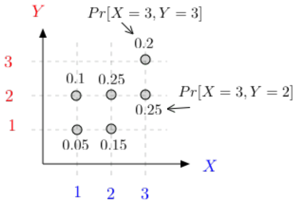


Marginal Distribution?

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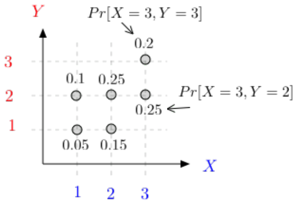


Marginal Distribution? Here is one.

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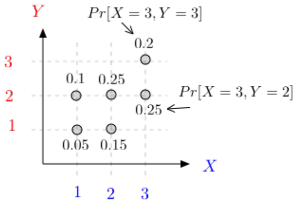


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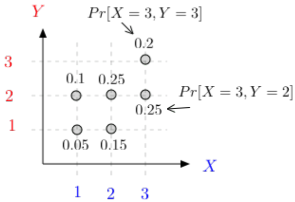


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The distribution of one of the variables.

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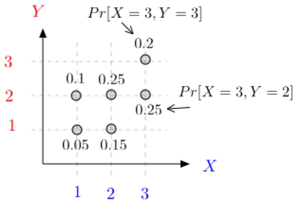


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$E[Y|X]$?

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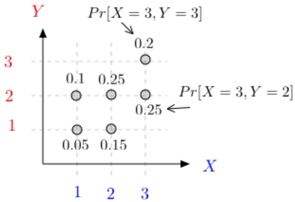


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$$E[Y|X=1] = (.03\times 1 + .2\times 2 + .21\times 3 + .02\times 5)/.44 = \frac{1.16}{.44}.$$

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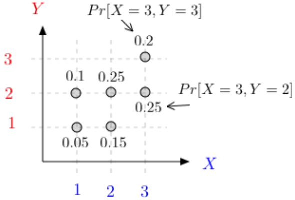


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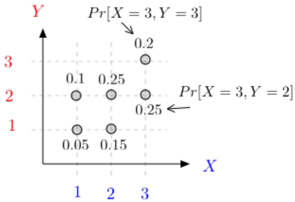


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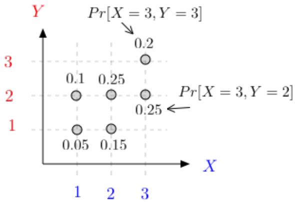


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$$E[Y|X=4] = (.1\times 1 + .03\times 2 + .03\times 3 + .02\times 5)/.18 = \frac{.35}{.18}.$$
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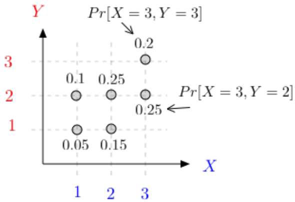


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$$E[Y|X=8] = (.02\times 1 + .02\times 2 + .01\times 3 + .01\times 5)/.06 = \frac{.10}{.06}.$$

$E[Y]$

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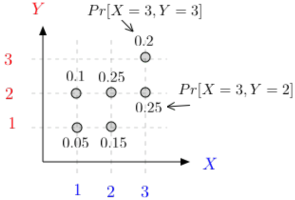


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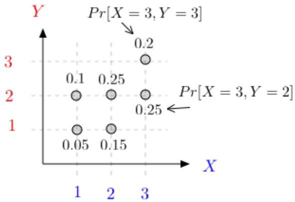


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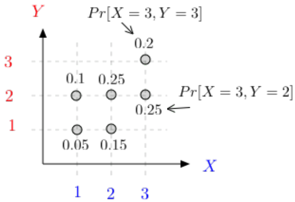


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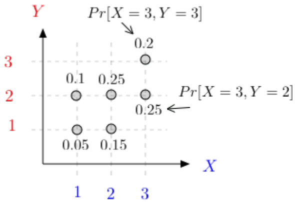


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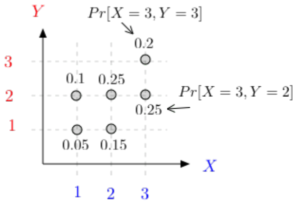


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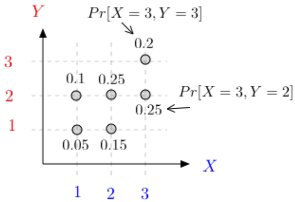


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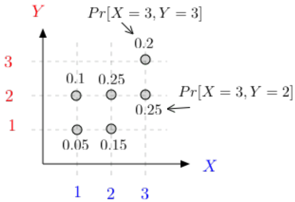


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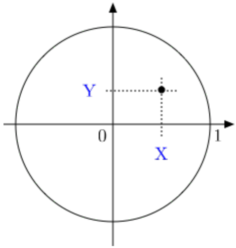

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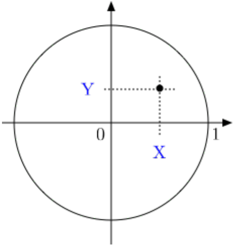


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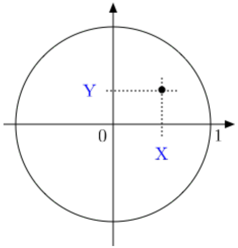


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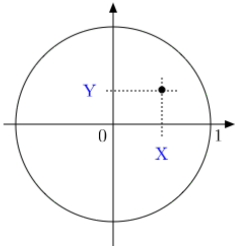


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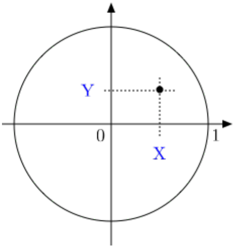


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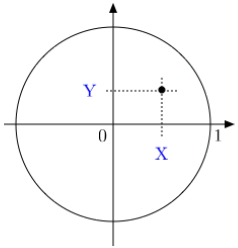


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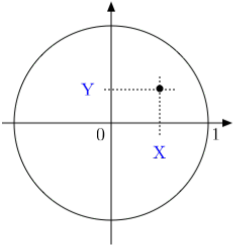


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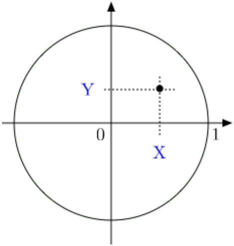


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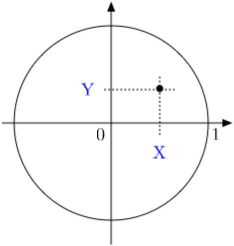


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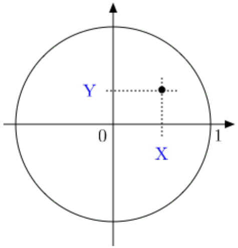


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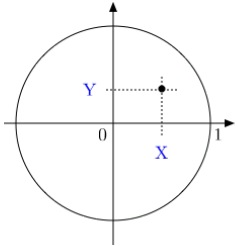

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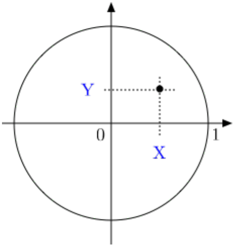


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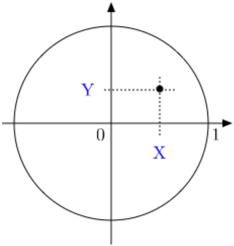


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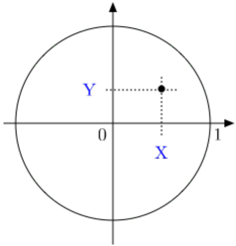


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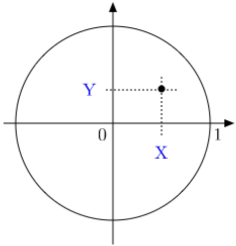


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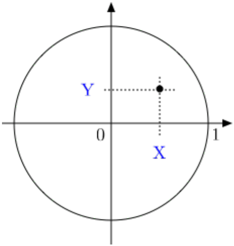


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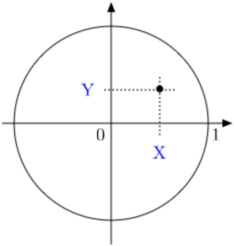


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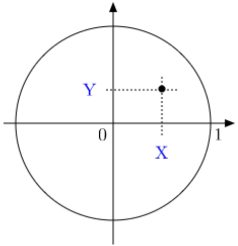


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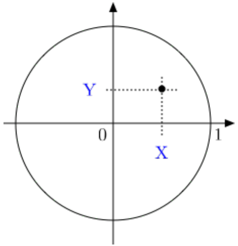


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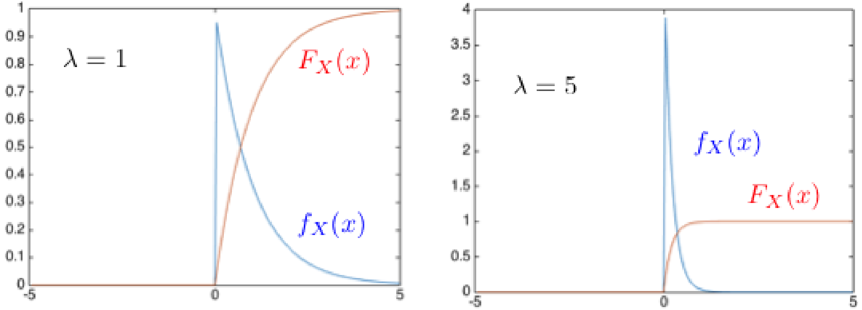

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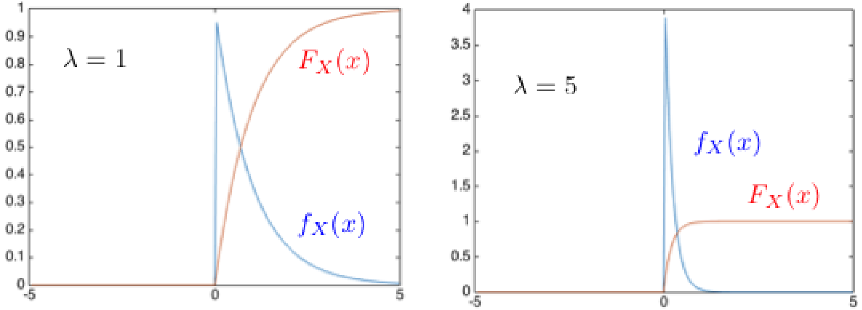


Note that $Pr[X > t] = e^{-\lambda t}$ for $t > 0$.

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Replacing $b$ by $b-a$ we see that, if $X = U[0,1]$, then $Y = a + (b-a)X$ is $U[a,b]$.

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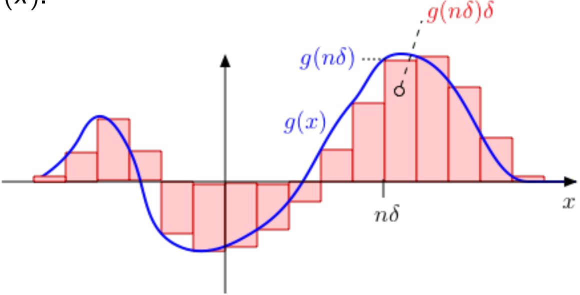

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Hence, $E[X] = \frac{1}{\lambda}$.

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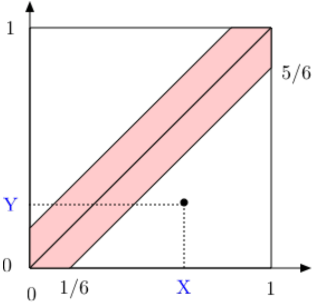

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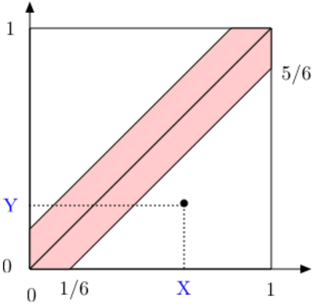


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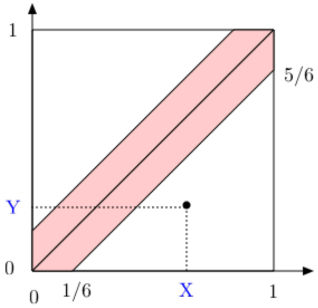


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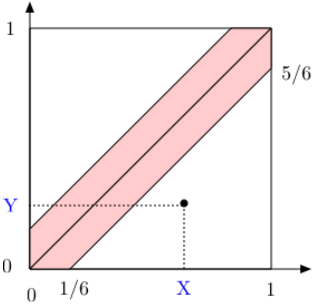


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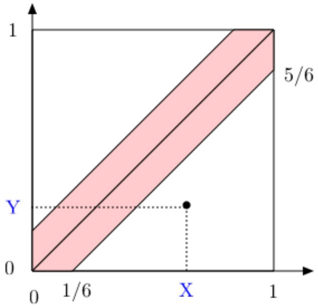


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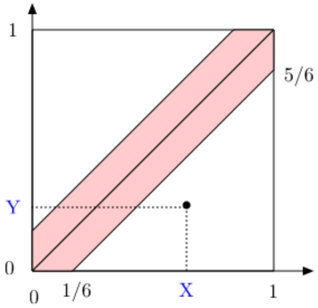


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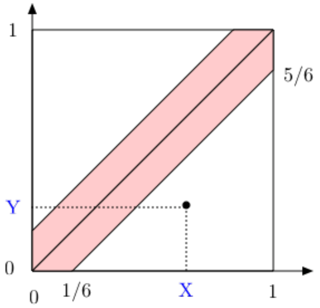


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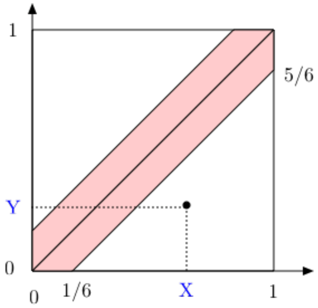


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Thus, $Pr[\text{meet}] = 1 - (\frac{5}{6})^2 =$

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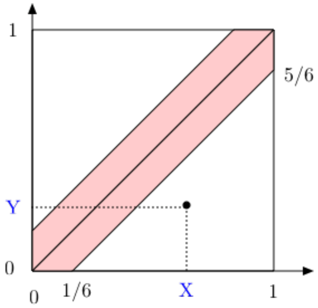


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The shaded area are the pairs where $|X - Y| < 1/6$, i.e., such that they meet.

The complement is the sum of two rectangles. When you put them together, they form a square with sides $5/6$.

Thus, $Pr[\text{meet}] = 1 - (\frac{5}{6})^2 = \frac{11}{36}$.

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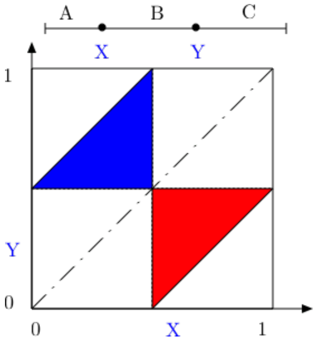

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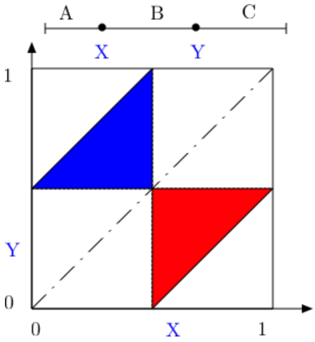


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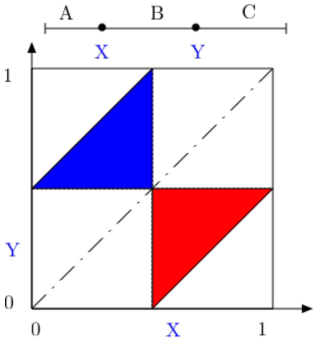


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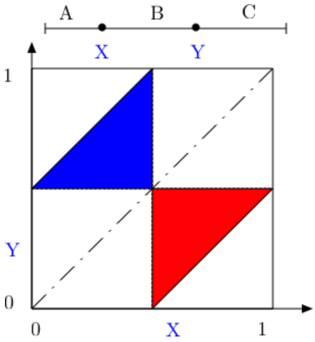


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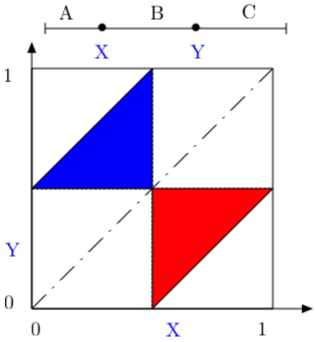


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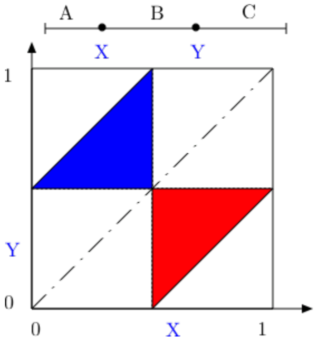


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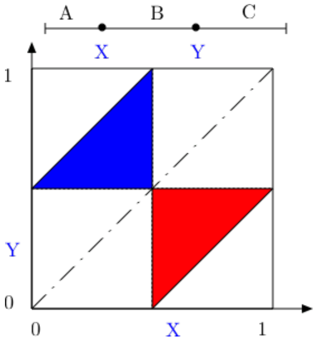


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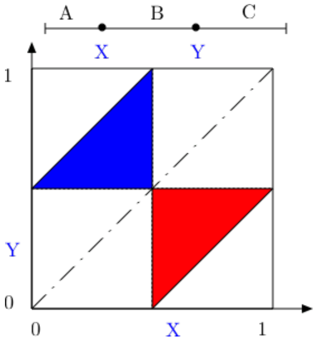


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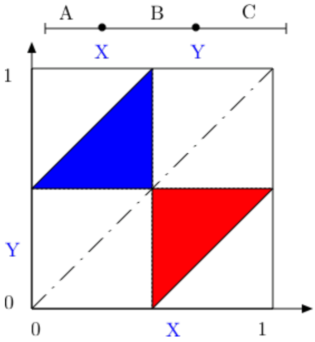


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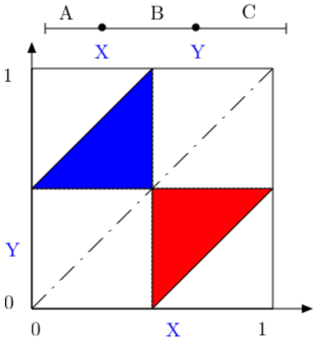


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Thus, $Pr[\text{make triangle}] = 1/4$.

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Hence,

$$E[Z] = A_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} = H(n).$$

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For instance, if $n = 16$, then $SNR(dB) \approx 112dB$.

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Indeed, $(1 - \frac{a}{N})^N \approx \exp\{-a\}$.

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