Today

Finish Euclid.

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Bijection/CRT/Isomorphism.

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Bijection/CRT/Isomorphism.

Review for Midterm.

Finding an inverse?

We showed how to efficiently tell if there is an inverse.

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Extend euclid to find inverse.

Euclid's GCD algorithm.

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(define (euclid x y)
 (if (= y 0)
     x
    (euclid  y (mod x y))))
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Computes the $\gcd(x, y)$ in $O(n)$ divisions.

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Computes the $\gcd(x, y)$ in $O(n)$ divisions.

For x and m, if $\gcd(x, m) = 1$ then x has an inverse modulo m.

Multiplicative Inverse.

GCD algorithm used to tell **if** there is a multiplicative inverse.

Multiplicative Inverse.

GCD algorithm used to tell **if** there is a multiplicative inverse.

How do we **find** a multiplicative inverse?

Extended GCD

Euclid's Extended GCD Theorem: For any x, y there are integers a, b such that

$$ax + by$$

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$ax + by = d$ where $d = gcd(x, y)$.

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"Make d out of sum of multiples of x and y."

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What is multiplicative inverse of x modulo m?

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$$ax + bm = 1$$
$$ax \equiv 1 - bm \equiv 1 \pmod{m}.$$

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So a multiplicative inverse of $x \pmod{m}$!!

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So a multiplicative inverse of $x \pmod{m}$!!
Example: For $x = 12$ and $y = 35$, $\gcd(12, 35) = 1$.

Extended GCD

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$$ax + by = d \qquad \text{where } d = gcd(x, y).$$

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So a multiplicative inverse of $x \pmod{m}$!!
Example: For $x = 12$ and $y = 35$, $\gcd(12, 35) = 1$.

$$(3)12 + (-1)35 = 1.$$

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So a multiplicative inverse of $x \pmod{m}$!!
Example: For $x = 12$ and $y = 35$, $\gcd(12, 35) = 1$.

$$(3)12 + (-1)35 = 1.$$

$a = 3$ and $b = -1$.

Extended GCD

Euclid's Extended GCD Theorem: For any x, y there are integers a, b such that

$$ax + by = d \qquad \text{where } d = gcd(x, y).$$

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What is multiplicative inverse of x modulo m?

By extended GCD theorem, when $\gcd(x, m) = 1$.

$$ax + bm = 1$$
$$ax \equiv 1 - bm \equiv 1 \pmod{m}.$$

So a multiplicative inverse of $x \pmod{m}$!!
Example: For $x = 12$ and $y = 35$, $\gcd(12, 35) = 1$.

$$(3)12 + (-1)35 = 1.$$

$a = 3$ and $b = -1$.
The multiplicative inverse of $12 \pmod{35}$ is 3.

Make *d* out of *x* and *y*..?

```
gcd(35,12)
```

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
```

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
```

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
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```

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How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.

Make d out of x and y..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
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```

How did gcd get 11 from 35 and 12?
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$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
$1 = 12 - (1)11$

Make d out of x and y..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
$1 = 12 - (1)11 = 12 - (1)(35 - (2)12)$
Get 11 from 35 and 12 and plugin....

Make d out of x and y..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
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But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
$1 = 12 - (1)11 = 12 - (1)(35 - (2)12) = (3)12 + (-1)35$
Get 11 from 35 and 12 and plugin.... Simplify.

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
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Get 11 from 35 and 12 and plugin.... Simplify.

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Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
$1 = 12 - (1)11 = 12 - (1)(35 - (2)12) = (3)12 + (-1)35$
Get 11 from 35 and 12 and plugin.... Simplify. $a = 3$ and $b = -1$.

Extended GCD Algorithm.

```
ext-gcd(x,y)
  if y = 0 then return(x, 1, 0)
     else
         (d, a, b) := ext-gcd(y, mod(x,y))
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Claim: Returns (d, a, b): $d = gcd(a, b)$ and $d = ax + by$.

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Example:

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Example:

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ext-gcd(35,12)
  ext-gcd(12, 11)
```

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Claim: Returns (d,a,b): $d = gcd(a,b)$ and $d = ax + by$.
Example: $a - \lfloor x/y \rfloor \cdot b =$

```
ext-gcd(35,12)
  ext-gcd(12, 11)
    ext-gcd(11, 1)
      ext-gcd(1,0)
      return (1,1,0) ;; 1 = (1)1 + (0) 0
```

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Claim: Returns (d,a,b): $d = gcd(a,b)$ and $d = ax + by$.
Example: $a - \lfloor x/y \rfloor \cdot b = 1 - \lfloor 11/1 \rfloor \cdot 0 = 1$

```
ext-gcd(35,12)
  ext-gcd(12, 11)
    ext-gcd(11, 1)
      ext-gcd(1,0)
      return (1,1,0) ;; 1 = (1)1 + (0) 0
    return (1,0,1)   ;; 1 = (0)11 + (1)1
```

Extended GCD Algorithm.

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ext-gcd(x,y)
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```

Claim: Returns (d,a,b): $d = gcd(a,b)$ and $d = ax + by$.
Example: $a - \lfloor x/y \rfloor \cdot b = 0 - \lfloor 12/11 \rfloor \cdot 1 = -1$

```
ext-gcd(35,12)
  ext-gcd(12, 11)
    ext-gcd(11, 1)
      ext-gcd(1,0)
      return (1,1,0) ;; 1 = (1)1 + (0) 0
    return (1,0,1)   ;; 1 = (0)11 + (1)1
  return (1,1,-1)    ;; 1 = (1)12 + (-1)11
```

Extended GCD Algorithm.

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ext-gcd(x,y)
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```

Claim: Returns (d,a,b): $d = gcd(a,b)$ and $d = ax + by$.
Example: $a - \lfloor x/y \rfloor \cdot b1 = \lfloor 35/12 \rfloor \cdot (-1) = 3$

```
    ext-gcd(35,12)
      ext-gcd(12, 11)
        ext-gcd(11, 1)
          ext-gcd(1,0)
          return (1,1,0) ;; 1 = (1)1 + (0) 0
        return (1,0,1)   ;; 1 = (0)11 + (1)1
      return (1,1,-1)    ;; 1 = (1)12 + (-1)11
   return (1,-1, 3)      ;; 1 = (-1)35 +(3)12
```

Extended GCD Algorithm.

```
ext-gcd(x,y)
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     else
         (d, a, b) := ext-gcd(y, mod(x,y))
         return (d, b, a - floor(x/y)  * b)
```

Claim: Returns (d, a, b): $d = gcd(a, b)$ and $d = ax + by$.
Example:

```
    ext-gcd(35,12)
      ext-gcd(12, 11)
        ext-gcd(11, 1)
          ext-gcd(1,0)
          return (1,1,0) ;; 1 = (1)1 + (0) 0
        return (1,0,1)   ;; 1 = (0)11 + (1)1
      return (1,1,-1)    ;; 1 = (1)12 + (-1)11
   return (1,-1, 3)      ;; 1 = (-1)35 +(3)12
```

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```
ext-gcd(x,y)
  if y = 0 then return(x, 1, 0)
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         (d, a, b) := ext-gcd(y, mod(x,y))
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ext-gcd(x,y)
  if y = 0 then return(x, 1, 0)
     else
         (d, a, b) := ext-gcd(y, mod(x,y))
         return (d, b, a - floor(x/y)  * b)
```

Theorem: Returns (d,a,b), where $d = gcd(a,b)$ and

$$d = ax + by.$$

Correctness.

Proof: Strong Induction.[1]

[1] Assume d is $gcd(x, y)$ by previous proof.

Correctness.

Proof: Strong Induction.[1]
Base: ext-gcd$(x, 0)$ returns $(d = x, 1, 0)$ with $x = (1)x + (0)y$.

[1] Assume d is $gcd(x, y)$ by previous proof.

Correctness.

Proof: Strong Induction.[1]
Base: ext-gcd$(x, 0)$ returns $(d = x, 1, 0)$ with $x = (1)x + (0)y$.

Induction Step: Returns (d, A, B) with $d = Ax + By$
Ind hyp: **ext-gcd**$(y, \mod (x, y))$ returns (d, a, b) with
$d = ay + b(\mod (x, y))$

[1] Assume d is $gcd(x, y)$ by previous proof.

Correctness.

Proof: Strong Induction.[1]
Base: ext-gcd($x,0$) returns ($d = x, 1, 0$) with $x = (1)x + (0)y$.

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ext-gcd(x,y) calls **ext-gcd**(y, $\mod(x,y)$) so

[1] Assume d is $gcd(x,y)$ by previous proof.

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Base: ext-gcd$(x, 0)$ returns $(d = x, 1, 0)$ with $x = (1)x + (0)y$.

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$d = ay + b(\mod(x, y))$

ext-gcd(x, y) calls **ext-gcd**$(y, \mod(x, y))$ so

$$d = ay + b \cdot (\mod(x, y))$$

[1] Assume d is $gcd(x, y)$ by previous proof.

Correctness.

Proof: Strong Induction.[1]
Base: ext-gcd$(x,0)$ returns $(d=x,1,0)$ with $x=(1)x+(0)y$.

Induction Step: Returns (d,A,B) with $d=Ax+By$
Ind hyp: **ext-gcd**$(y,\ \bmod(x,y))$ returns (d,a,b) with
$d=ay+b(\bmod(x,y))$

ext-gcd(x,y) calls **ext-gcd**$(y,\ \bmod(x,y))$ so

$$\begin{aligned} d &= ay+b\cdot(\bmod(x,y)) \\ &= ay+b\cdot(x-\lfloor\frac{x}{y}\rfloor y) \end{aligned}$$

[1] Assume d is $gcd(x,y)$ by previous proof.

Correctness.

Proof: Strong Induction.[1]
Base: ext-gcd$(x,0)$ returns $(d=x,1,0)$ with $x=(1)x+(0)y$.

Induction Step: Returns (d,A,B) with $d=Ax+By$
Ind hyp: **ext-gcd**$(y, \mod(x,y))$ returns (d,a,b) with
$d=ay+b(\mod(x,y))$

ext-gcd(x,y) calls **ext-gcd**$(y, \mod(x,y))$ so

$$\begin{aligned} d &= ay+b\cdot(\mod(x,y)) \\ &= ay+b\cdot(x-\lfloor\frac{x}{y}\rfloor y) \\ &= bx+(a-\lfloor\frac{x}{y}\rfloor\cdot b)y \end{aligned}$$

[1] Assume d is $gcd(x,y)$ by previous proof.

Correctness.

Proof: Strong Induction.[1]
Base: ext-gcd$(x,0)$ returns $(d = x, 1, 0)$ with $x = (1)x + (0)y$.

Induction Step: Returns (d, A, B) with $d = Ax + By$
Ind hyp: **ext-gcd**$(y, \mod(x,y))$ returns (d,a,b) with
$d = ay + b(\mod(x,y))$

ext-gcd(x,y) calls **ext-gcd**$(y, \mod(x,y))$ so

$$\begin{aligned} d &= ay + b \cdot (\mod(x,y)) \\ &= ay + b \cdot (x - \lfloor \frac{x}{y} \rfloor y) \\ &= bx + (a - \lfloor \frac{x}{y} \rfloor \cdot b)y \end{aligned}$$

And ext-gcd returns $(d, b, (a - \lfloor \frac{x}{y} \rfloor \cdot b))$ so theorem holds!

[1] Assume d is $gcd(x,y)$ by previous proof.

Correctness.

Proof: Strong Induction.[1]
Base: ext-gcd$(x, 0)$ returns $(d = x, 1, 0)$ with $x = (1)x + (0)y$.

Induction Step: Returns (d, A, B) with $d = Ax + By$
Ind hyp: **ext-gcd**$(y, \mod(x, y))$ returns (d, a, b) with
$d = ay + b(\mod(x, y))$

ext-gcd(x, y) calls **ext-gcd**$(y, \mod(x, y))$ so

$$\begin{aligned} d &= ay + b \cdot (\mod(x, y)) \\ &= ay + b \cdot (x - \lfloor \frac{x}{y} \rfloor y) \\ &= bx + (a - \lfloor \frac{x}{y} \rfloor \cdot b)y \end{aligned}$$

And ext-gcd returns $(d, b, (a - \lfloor \frac{x}{y} \rfloor \cdot b))$ so theorem holds! □

[1] Assume d is $gcd(x, y)$ by previous proof.

Review Proof: step.

```
ext-gcd(x,y)
  if y = 0 then return(x, 1, 0)
     else
         (d, a, b) := ext-gcd(y, mod(x,y))
         return (d, b, a - floor(x/y)  * b)
```

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Internet Security: Next Week!

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$x = 5 \pmod 7$ and $x = 3 \pmod 5$.

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A bit slow for large values.

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Only solution? If not, two solutions, x and y.

Simple Chinese Remainder Theorem.

Find $x = a \pmod{m}$ and $x = b \pmod{n}$ where gcd(m,n)=1.

CRT Thm: Unique solution $\pmod{mn}$.
Proof:
Consider $u = n(n^{-1} \pmod{m})$.
$u = 0 \pmod{n}$ $\qquad u = 1 \pmod{m}$

Consider $v = m(m^{-1} \pmod{n})$.
$v = 1 \pmod{n}$ $\qquad v = 0 \pmod{m}$

Let $x = au + bv$.
$x = a \pmod{m}$ since $bv = 0 \pmod{m}$ and $au = a \pmod{m}$
$x = b \pmod{n}$ since $au = 0 \pmod{n}$ and $bv = b \pmod{n}$

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$(x - y) \equiv 0 \pmod{m}$ and $(x - y) \equiv 0 \pmod{n}$.

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Only solution? If not, two solutions, x and y.
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$\implies (x - y)$ is multiple of m and n since gcd(m,n)=1.

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Thus, only one solution modulo mn.

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Thus, only one solution modulo mn. $\square$

Midterm Review

Now...

First there was logic...

A statement is a true or false.

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A statement is a true or false.
Statements?

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Statements?

$3 = 4 - 1$?

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Statements?

$3 = 4 - 1$? Statement!

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Statements?
$3 = 4 - 1$? Statement!
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Statements?

$3 = 4 - 1$? Statement!

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$n = 3$? Not a statement...but a predicate.

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Statements?

$3 = 4 - 1$? Statement!
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Predicate: Statement with free variable(s).

First there was logic...

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Example: $x = 3$

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$n > 3$?

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Predicate: Statement with free variable(s).

Example: $x = 3$

Given a value for x, becomes a statement.

Predicate?

$n > 3$? Predicate: $P(n)$!

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Quantifiers:

$(\forall x)\ P(x)$. For every x, $P(x)$ is true.

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Connecting Statements

$A \wedge B$, $A \vee B$, $\neg A$.

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You got this!

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Propositional Expressions and Logical Equivalence

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Propositional Expressions and Logical Equivalence

$(A \implies B) \equiv (\neg A \vee B)$

Connecting Statements

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Propositional Expressions and Logical Equivalence

$(A \implies B) \equiv (\neg A \vee B)$
$\neg(A \vee B) \equiv (\neg A \wedge \neg B)$

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Proofs: truth table or manipulation of known formulas.

Connecting Statements

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Proofs: truth table or manipulation of known formulas.

$(\forall x)(P(x) \wedge Q(x)) \equiv (\forall x)P(x) \wedge (\forall x)Q(x)$

..and then proofs...

Direct: $P \implies Q$

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Direct: $P \implies Q$

Example: a is even $\implies a^2$ is even.

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Approach: What is even?

..and then proofs...

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Approach: What is even? $a = 2k$

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What is even?

$a^2 = 2(2k^2)$

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Example: a^2 is odd $\implies$ a is odd.

Contrapositive: a is even $\implies$ a^2 is even.

Contradiction: P

$\neg P \implies$ **false**

..and then proofs...

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$\neg P \implies R \wedge \neg R$

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Useful for prove something does not exist:

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Useful for prove something does not exist:

Example: rational representation of $\sqrt{2}$

..and then proofs...

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Integers closed under multiplication!

a^2 is even.

Contrapositive: $P \implies Q$ or $\neg Q \implies \neg P$.

Example: a^2 is odd $\implies a$ is odd.

Contrapositive: a is even $\implies a^2$ is even.

Contradiction: P

$\neg P \implies$ **false**

$\neg P \implies R \wedge \neg R$

Useful for prove something does not exist:

Example: rational representation of $\sqrt{2}$ does not exist.

..and then proofs...

Direct: $P \implies Q$

Example: a is even $\implies a^2$ is even.

Approach: What is even? $a = 2k$

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Example: rational representation of $\sqrt{2}$ does not exist.

Example: finite set of primes does not exist.

Example: rogue couple does not exist.

...jumping forward..

Contradiction in induction:

...jumping forward..

Contradiction in induction:
contradict place where induction step doesn’t hold.

...jumping forward..

Contradiction in induction:
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Well Ordering Principle.

...jumping forward..

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Well Ordering Principle.
 Stable Marriage:

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first day where women does not improve.

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...and then induction...

$$P(0) \wedge ((\forall n)(P(n) \implies P(n+1) \equiv (\forall n \in N)\ P(n).$$

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Thm: For all $n \geq 1$, $8 | 3^{2n} - 1$.

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Induction on n.

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$$3^{2n+2}-1 = 9(3^{2n})-1$$

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Induction on n.

Base: $8 | 3^2 - 1$.

Induction Hypothesis: Assume $P(n)$: True for some n.
$(3^{2n} - 1 = 8d)$

Induction Step: Prove $P(n+1)$

$3^{2n+2} - 1 = 9(3^{2n}) - 1$ (by induction hypothesis)

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$$3^{2n+2} - 1 = 9(3^{2n}) - 1 \text{ (by induction hypothesis)}$$
$$= 9(8d + 1) - 1$$

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$$\begin{aligned} 3^{2n+2}-1 &= 9(3^{2n})-1 \quad \text{(by induction hypothesis)} \\ &= 9(8d+1)-1 \\ &= 72d+8 \end{aligned}$$

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Divisible by 8.

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Divisible by 8.

□

Stable Marriage: a study in definitions and WOP.

n-men, n-women.

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n-men, *n*-women.

Each person has completely ordered preference list

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Pairing with no rogue couples.

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Rogue Couple in a pairing.
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Pairing with no rogue couples.

Does stable pairing exist?

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Does stable pairing exist?

No, for roommates problem.

TMA.

Traditional Marriage Algorithm:

TMA.

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Each Day:

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"Propose and Reject."

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Not rogue couple!

Optimality/Pessimal

Optimal partner if best partner in any stable pairing.

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M' likes W at least as much as optimal partner.

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Optimal partner if best partner in any stable pairing.
 Not necessarily first in list.
 Possibly no stable pairing with that partner.

Man-optimal pairing is pairing where every man gets optimal partner.

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 First man M to lose optimal partner.
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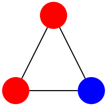

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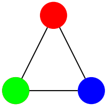

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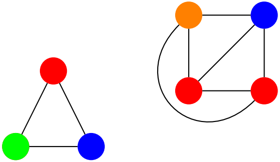

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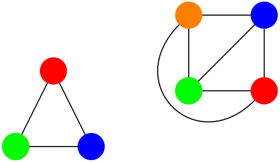

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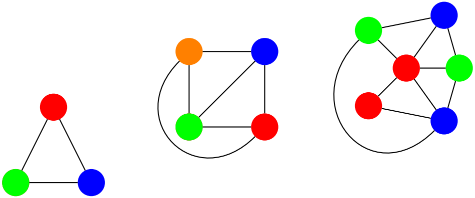

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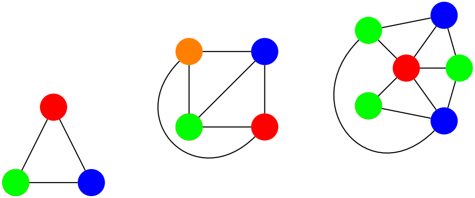

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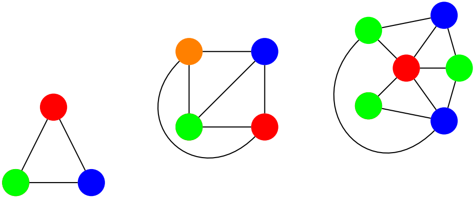

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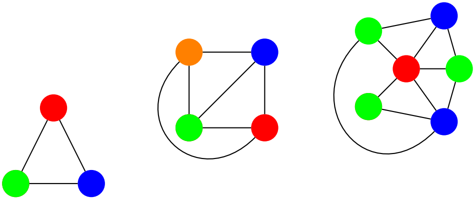

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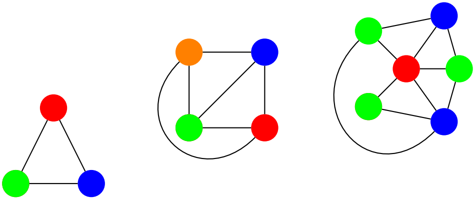

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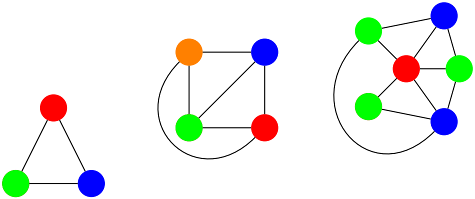

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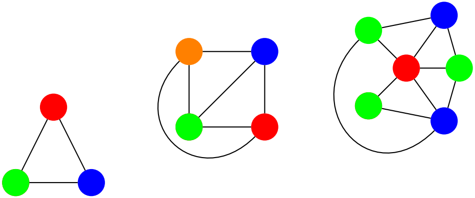

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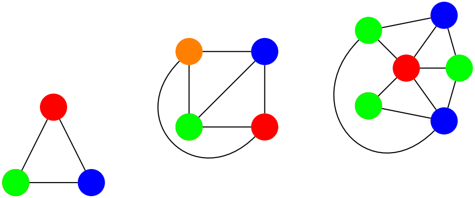

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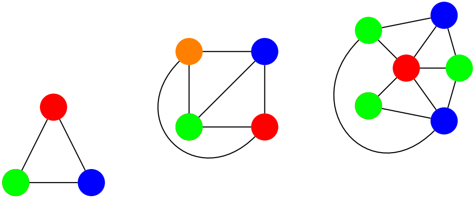

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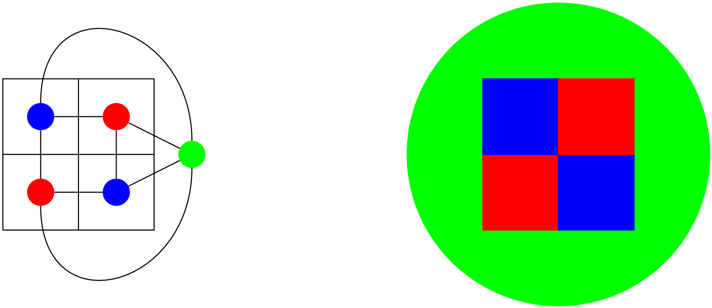
Planar graph coloring $\equiv$ map coloring.



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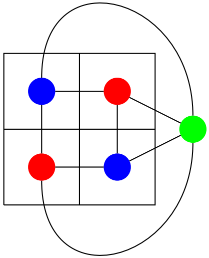

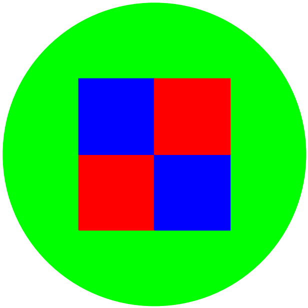

Four color theorem is about planar graphs!

Six color theorem.

Theorem: Every planar graph can be colored with six colors.

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Preliminary Observation: Connected components of vertices with two colors in a legal coloring can switch colors.

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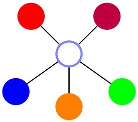

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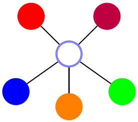

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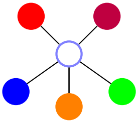

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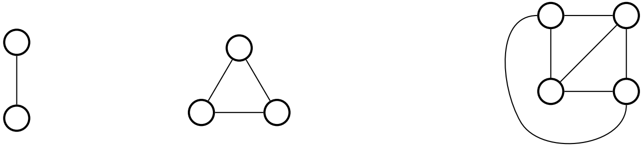

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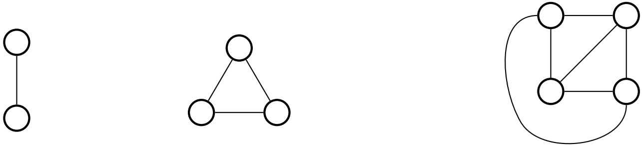

K_n, $|V| = n$

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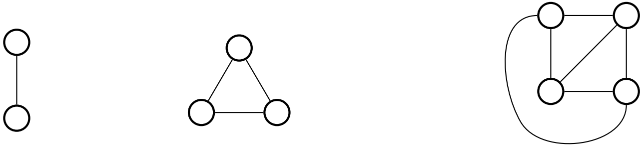

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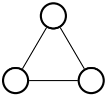

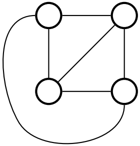

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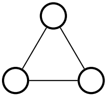
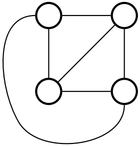

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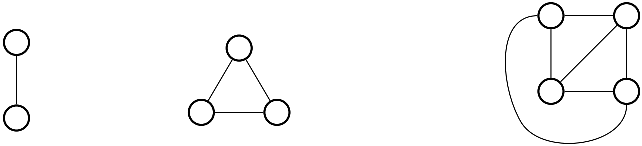

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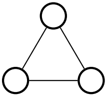

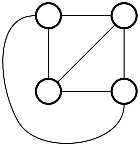

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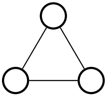
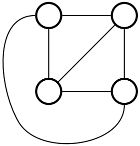

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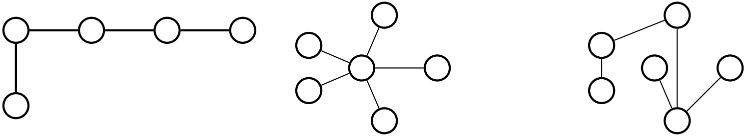
degree of vertex? $|V| - 1$.

Very connected.
Lots of edges: $n(n-1)/2$.

Trees.



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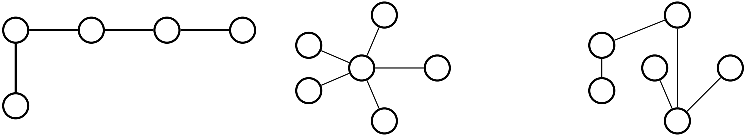

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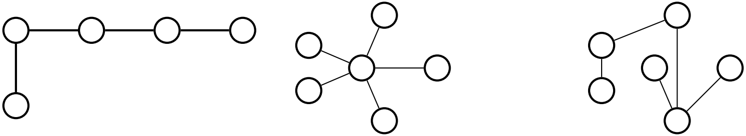

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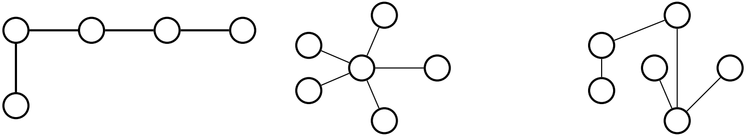

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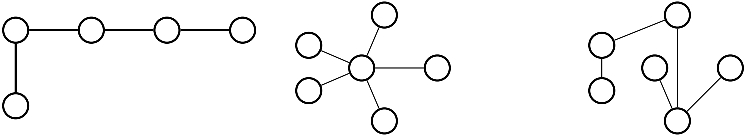

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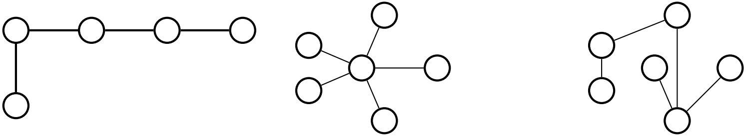

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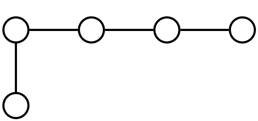 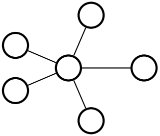 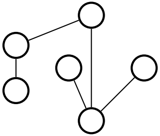

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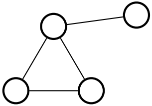
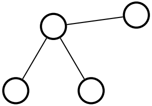
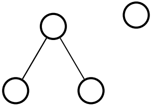
To tree or not to tree!

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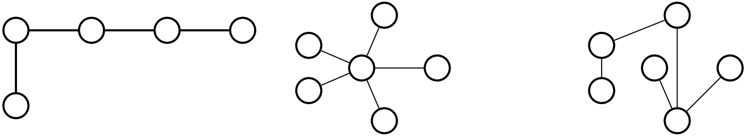

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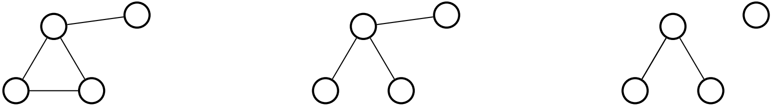

Minimally connected, minimum number of edges to connect.

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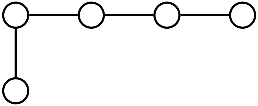
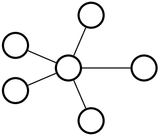
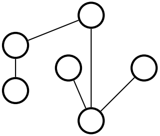

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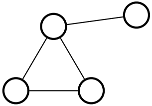
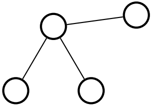
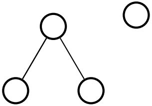

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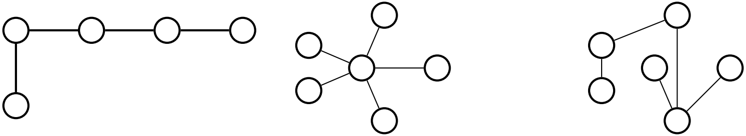

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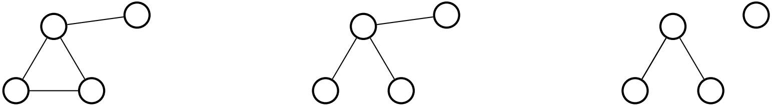

Minimally connected, minimum number of edges to connect.

Property:
Can remove a single node and break into components of size at most $|V|/2$.

Hypercube

Hypercubes.

Hypercube

Hypercubes. Really connected.

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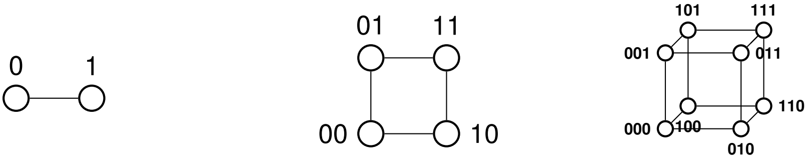

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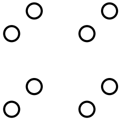

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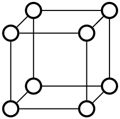

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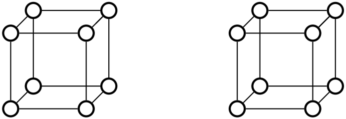

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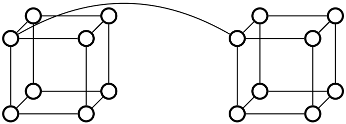

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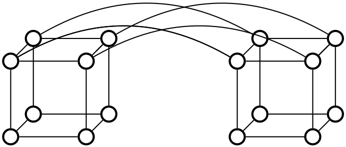

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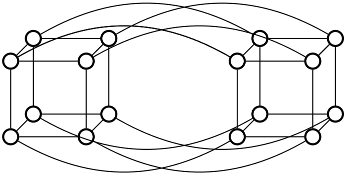

Hypercube:properties

Rudrata Cycle: cycle that visits every node.

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Good communication network!

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$58 + 32 = 90 = 6 \pmod{7}$

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$58 + 32 = 2 + 4 = 6 \pmod{7}$
$58 + 32 = 2 + -3 = -1 = 6 \pmod{7}$

...Modular Arithmetic...

Arithmetic modulo m.
Elements of equivalence classes of integers.
$\{0,\ldots,m-1\}$
and integer $i \equiv a \pmod{m}$
if $i = a + km$ for integer k.
or if the remainder of i divided by m is a.

Can do calculations by taking remainders
at the beginning,
in the middle
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Additive inverses are intuitively negative numbers.

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$3^{-1} \pmod 7$?

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Modular Arithmetic Inverses and GCD

x has inverse modulo m if and only if $gcd(x, m) = 1$.

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$$7(0) + 60(1) = 60$$

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Confirm: $-119 + 120 = 1$

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Confirm: $-119 + 120 = 1$

$d = e^{-1} = -17 = 43 = \pmod{60}$

Midterm format

Time: 120 minutes.

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Some short answers.

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Get at ideas that you learned.

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Know material well:

Midterm format

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Know material well: fast,

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Get at ideas that you learned.
Know material well: fast, correct.

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Get at ideas that you learned.
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Midterm format

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Proofs, algorithms,

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Know material well:	fast, correct.
Know material medium:	slower, less correct.
Know material not so well:	Uh oh.

Some longer questions.
Proofs, algorithms, properties.

Midterm format

Time: 120 minutes.

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See piazza for more resources.
E.g., TA videos for past exams.

Wrapup.

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Other issues....

Wrapup.

Other issues....
admin@eecs70.org

Wrapup.

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Private message on piazza.

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