## Today.

Couple of more induction proofs.

Stable Marriage.



## Strengthening: need to...

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

Base: $P(1)$. $1 \leq 2$.
Ind Step: $\sum_{i=1}^{k} \frac{1}{i^2} \leq 2$.

$\sum_{i=1}^{k+1} \frac{1}{i^2}$
$= \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2}$.
$\leq 2 + \frac{1}{(k+1)^2}$

Uh oh?

Hmmm... It better be that any sum is *strictly less than* 2.

How much less? At least by $\frac{1}{(k+1)^2}$ for $S_k$.

"$S_k \leq 2 - \frac{1}{(k+1)^2}$" $\implies$ "$S_{k+1} \leq 2$"

Induction step works! No! Not the same statement!!!!
Need to prove "$S_{k+1} \leq 2 - \frac{1}{(k+2)^2}$".

Darn!!!



## Strenthening: how?

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n)$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)
**Proof:**
Ind hyp: $P(k)$ — "$S_k \leq 2 - f(k)$"
Prove: $P(k+1)$ – "$S_{k+1} \leq 2 - f(k+1)$"

$S(k+1) = S_k + \frac{1}{(k+1)^2}$
$\leq 2 - f(k) + \frac{1}{(k+1)^2}$ By ind. hyp.

Choose $f(k+1) \leq f(k) - \frac{1}{(k+1)^2}$.
$\implies S(k+1) \leq 2 - f(k+1)$.

Can you?
Subtracting off a quadratically decreasing function every time.
Maybe a linearly decreasing function to keep positive?
Try $f(k) = \frac{1}{k}$

$\frac{1}{k+1} \leq \frac{1}{k} - \frac{1}{(k+1)^2}$?

$1 \leq \frac{k+1}{k} - \frac{1}{k+1}$ Multiplied by $k+1$.
$1 \leq 1 + (\frac{1}{k} - \frac{1}{k+1})$ Some math. So yes!

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$.



## Stable Marriage Problem

- Small town with $n$ boys and $n$ girls.
- Each girl has a ranked preference list of boys.
- Each boy has a ranked preference list of girls.

How should they be matched?



## Count the ways..

- Maximize total satisfaction.
- Maximize number of first choices.
- Maximize worse off.
- Minimize difference between preference ranks.



## The best laid plans..

Consider the couples..

- Jennifer and Brad
- Angelina and Billy-Bob

Brad prefers Angelina to Jennifer.

Angelina prefers Brad to BillyBob.

Uh..oh.

## So..

Produce a pairing where there is no running off!

**Definition:** A **pairing** is disjoint set of $n$ boy-girl pairs.

Example: A pairing $S = \{(Brad, Jen); (BillyBob, Angelina)\}$.

**Definition:** A **rogue couple** $b, g^*$ for a pairing $S$:
$b$ and $g^*$ prefer each other to their partners in $S$

Example: Brad and Angelina are a rogue couple in $S$.



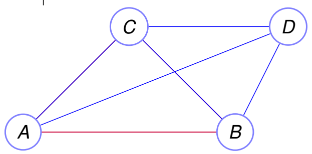
## A stable pairing??

Given a set of preferences.

Is there a stable pairing?
How does one find it?

Consider a single gender version: stable roommates.

| | | | |
|---|---|---|---|
| A | B | C | D |
| B | C | A | D |
| C | A | B | D |
| D | A | B | C |






## The Traditional Marriage Algorithm.

Each Day:

1. Each boy **proposes** to his favorite girl on his list.
2. Each girl rejects all but her favorite proposer (whom she puts on a string.)
3. Rejected boy crosses rejecting girl off his list.

Stop when each girl gets exactly one proposal.
Does this terminate?

...produce a pairing?

....a stable pairing?

Do boys or girls do "better"?



## Example.

| Boys | | | |
|---|---|---|---|
| A | ~~1~~ | 2 | 3 |
| B | ~~1~~ | ~~2~~ | 3 |
| C | ~~2~~ | 1 | 3 |

| Girls | | | |
|---|---|---|---|
| 1 | C | A | B |
| 2 | A | B | C |
| 3 | A | C | B |

| | Day 1 | Day 2 | Day 3 | Day 4 | Day 5 |
|---|---|---|---|---|---|
| 1 | A, ~~B~~ | A | ~~A~~ , C | C | C |
| 2 | C | B, ~~C~~ | B | A,~~B~~ | A |
| 3 | | | | | B |



## Termination.

Every non-terminated day a boy **crossed** an item off the list.

Total size of lists? $n$ boys, $n$ length list. $n^2$

Terminates in at most $n^2 + 1$ steps!



## It gets better every day for girls..

**Improvement Lemma: It just gets better for girls.**
If on day $t$ a girl $g$ has a boy $b$ on a string,
any boy, $b'$, on $g$'s string for any day $t' > t$
is at least as good as $b$.

**Proof:**
$P(k)$- - "boy on $g$'s string is at least as good as $b$ on day $t + k$"

$P(0)$– true. Girl has $b$ on string.

Assume $P(k)$. Let $b'$ be boy **on string** on day $t + k$.

On day $t + k + 1$, boy $b'$ comes back.
Girl can choose $b'$, or do better with another boy, $b''$

That is, $b \leq b'$ by induction hypothesis.
And $b''$ is better than $b'$ by algorithm.
$\implies$ Girl does at least as well as with $b$.

$P(k) \implies P(k+1)$. And by principle of induction. □

## Pairing when done.

**Lemma:** Every boy is matched at end.

**Proof:**
If not, a boy $b$ must have been rejected $n$ times.

Every girl has been proposed to by $b$,
and Improvement lemma

$\implies$ each girl has a boy on a string.

and each boy is on at most one string.

$n$ girls and $n$ boys. Same number of each.

$\implies$ $b$ must be on some girl's string!

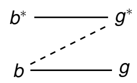
Contradiction. □



## Pairing is Stable.

**Lemma:** There is no rogue couple for the pairing formed by traditional marriage algorithm.

**Proof:**
Assume there is a rogue couple; $(b, g^*)$




$b$ likes $g^*$ more than $g$.

$g^*$ likes $b$ more than $b^*$.

Boy $b$ proposes to $g^*$ before proposing to $g$.

So $g^*$ rejected $b$ (since he moved on)

By improvement lemma, $g^*$ likes $b^*$ better than $b$.

Contradiction! □



## Good for boys? girls?

Is the TMA better for boys? for girls?

**Definition:** A **pairing is $x$-optimal** if $x's$ partner
is its best partner in any stable pairing.

**Definition:** A **pairing is $x$-pessimal** if $x's$ partner
is its worst partner in any stable pairing.

**Definition:** A **pairing is boy optimal** if it is $x$-optimal for **all** boys $x$.

..and so on for boy pessimal, girl optimal, girl pessimal.

Claim: The optimal partner for a boy must be first in his preference list.

True? False? False!

Subtlety here: Best partner in any stable pairing.
As well as you can be in a globally stable solution!

Question: Is there a boy or girl optimal pairing?
Is it possible:
$b$-optimal pairing different from the $b'$-optimal pairing!
Yes? No?



## TMA is optimal!

For boys? For girls?

**Theorem:** TMA produces a boy-optimal pairing.

**Proof:**
Assume not: there are boys who do not get their optimal girl.

Let $t$ be first day a boy $b$ gets rejected
by his optimal girl $g$ who he is paired with
in stable pairing $S$.

$b^*$ - knocks $b$ off of $g$'s string on day $t$ $\implies$ $g$ prefers $b^*$ to $b$

By choice of $t$, $b^*$ prefers $g$ to optimal girl.

$\implies$ $b^*$ prefers $g$ to his partner $g^*$ in $S$.

Rogue couple for $S$.
So $S$ is not a stable pairing. Contradiction. □

Notes: S - stable. $(b^*, g^*) \in S$. But $(b^*, g)$ is rogue couple!

Used Well-Ordering principle...Induction.



## How about for girls?

**Theorem:** TMA produces girl-pessimal pairing.

$T$ – pairing produced by TMA.

$S$ – worse stable pairing for girl $g$.

In $T$, $(g, b)$ is pair.

In $S$, $(g, b^*)$ is pair.

$g$ likes $b^*$ less than she likes $b$.

$T$ is boy optimal, so $b$ likes $g$ more than his partner in $S$.

$(g, b)$ is Rogue couple for $S$

$S$ is not stable.

Contradiction. □

Notes: Not really induction.
Structural statement: Boy optimality $\implies$ Girl pessimality.



## Quick Questions.

How does one make it better for girls?

SMA - stable marriage algorithm. One side proposes.
TMA - boys propose.
Girls could propose. $\implies$ optimal for girls.

## Residency Matching..

The method was used to match residents to hospitals.

Hospital optimal....

..until 1990's...Resident optimal.

Another variation: couples.



## Don't go!

Summary.

Link