

Today.

Principle of Induction.

$$P(0) \wedge (\forall n \in \mathbb{N})P(n) \implies P(n+1)$$

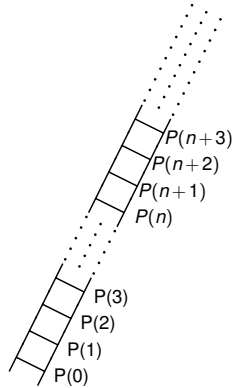
And we get...

$$(\forall n \in \mathbb{N})P(n).$$

...Yes for 0, and we can conclude Yes for 1...
and we can conclude Yes for 2.....

1/25

Climb an infinite ladder?



$$P(0) \implies P(1) \implies P(2) \implies P(3) \dots$$

$$\forall k, P(k) \implies P(k+1)$$

$$(\forall n \in \mathbb{N})P(n)$$

Your favorite example of forever..or the natural numbers...

2/25

Another Induction Proof.

Theorem: For every $n \in \mathbb{N}$, $n^3 - n$ is divisible by 3. ($3 | (n^3 - n)$).

Proof: By induction.

Base Case: $P(0)$ is " $(0^3) - 0$ " is divisible by 3. Yes!

Induction Step: $(\forall k \in \mathbb{N}), P(k) \implies P(k+1)$

Induction Hypothesis: $k^3 - k$ is divisible by 3.

or $k^3 - k = 3q$ for some integer q .

$$\begin{aligned} (k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - (k+1) \\ &= k^3 + 3k^2 + 2k \\ &= (k^3 - k) + 3k^2 + 3k \quad \text{Subtract/add } k \\ &= 3q + 3(k^2 + k) \quad \text{Induction Hyp. Factor.} \\ &= 3(q + k^2 + k) \quad \text{(Un)Distributive + over } \times \end{aligned}$$

Or $(k+1)^3 - (k+1) = 3(q + k^2 + k)$.

$(q + k^2 + k)$ is integer (closed under addition and multiplication).

$\implies (k+1)^3 - (k+1)$ is divisible by 3.

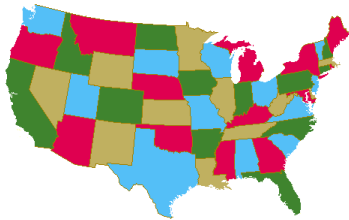
Thus, $(\forall k \in \mathbb{N})P(k) \implies P(k+1)$

Thus, theorem holds by induction. □

3/25

Four Color Theorem.

Theorem: Any map can be colored so that those regions that share an edge have different colors.



Check Out: "Four corners".

States connected at a point, can have same color.

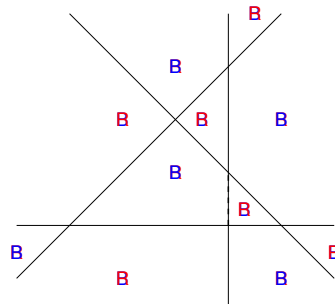
(Couldn't find a map where they did though.)

Quick Test: Which states? Utah. Colorado. New Mexico. Arizona.

4/25

Two color theorem: example.

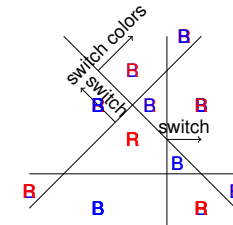
Any map formed by dividing the plane into regions by drawing straight lines can be properly colored with two colors.



Fact: Swapping red and blue gives another valid colors.

5/25

Two color theorem: proof illustration.



Base Case.

1. Add line.

2. Get inherited color for split regions

3. Switch on one side of new line.

(Fixes conflicts along line, and makes no new ones.)

Algorithm gives $P(k) \implies P(k+1)$. □

6/25

Strengthening Induction Hypothesis.

Theorem: The sum of the first n odd numbers is a perfect square.

Theorem: The sum of the first n odd numbers is n^2 .

k th odd number is $2(k-1)+1$.

Base Case 1 (first odd number) is 1^2 .

Induction Hypothesis Sum of first k odds is perfect square $a^2 = k^2$.

Induction Step 1. The $(k+1)$ st odd number is $2k+1$.

2. Sum of the first $k+1$ odds is

$$a^2 + 2k + 1 = k^2 + 2k + 1$$

????

3. $k^2 + 2k + 1 = (k+1)^2$

... $P(k+1)$!

□

7/25

Hole in center?

Theorem: Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.

Proof:

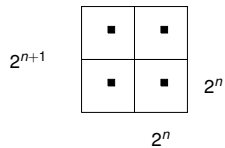
Base case: A single tile works fine.

The hole is adjacent to the center of the 2×2 square.

Induction Hypothesis:

Any $2^n \times 2^n$ square can be tiled with a hole at the center.

$$2^{n+1}$$



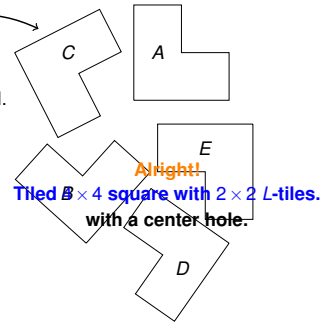
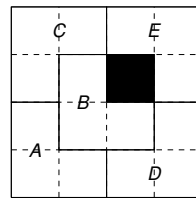
What to do now???

10/25

Tiling Cory Hall Courtyard.

Use these L-tiles.

To Tile this 4×4 courtyard.



Tiled 8×4 square with 2×2 L-tiles.
with a center hole.

Can we tile any $2^n \times 2^n$ with L-tiles (with a hole) for every n !

8/25

Hole can be anywhere!

Theorem: Can tile the $2^n \times 2^n$ to leave a hole adjacent *anywhere*.

Better theorem ...better induction hypothesis!

Base case: Sure. A tile is fine.



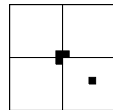
Flipping the orientation can leave hole anywhere.



Induction Hypothesis:

"Any $2^n \times 2^n$ square can be tiled with a hole **anywhere**."

Consider $2^{n+1} \times 2^{n+1}$ square.



Use induction hypothesis in each.

Use L-tile and ... we are done.

11/25

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Proof: The remainder of 2^{2n} divided by 3 is 1.

Base case: true for $k=0$. $2^0 = 1$

Ind Hyp: $2^{2k} = 3a+1$ for integer a .

$$\begin{aligned} 2^{2(k+1)} &= 2^{2k} * 2^2 \\ &= 4 * 2^{2k} \\ &= 4 * (3a+1) \\ &= 12a+3+1 \\ &= 3(4a+1)+1 \end{aligned}$$

a integer $\implies (4a+1)$ is an integer.

□

9/25

Strong Induction.

Theorem: Every natural number $n > 1$ can be written as a (possibly trivial) product of primes.

Definition: A prime n has exactly 2 factors 1 and n .

Base Case: $n=2$.

Induction Step:

$P(n)$ = " n can be written as a product of primes."

Either $n+1$ is a prime or $n+1 = a \cdot b$ where $1 < a, b < n+1$.

$P(n)$ says nothing about a, b !

Strong Induction Principle: If $P(0)$ and

$$(\forall k \in \mathbb{N})(P(0) \wedge \dots \wedge P(k)) \implies P(k+1),$$

then $(\forall k \in \mathbb{N})(P(k))$.

$$P(0) \implies P(1) \implies P(2) \implies P(3) \implies \dots$$

Strong induction hypothesis: " a and b are products of primes"

\implies " $n+1 = a \cdot b = (\text{factorization of } a)(\text{factorization of } b)$ "

$n+1$ can be written as the product of the prime factors!

□

12/25

Induction \implies Strong Induction.

Let $Q(k) = P(0) \wedge P(1) \cdots P(k)$.

By the induction principle:

"If $Q(0)$, and $(\forall k \in \mathbb{N})(Q(k) \implies Q(k+1))$ then $(\forall k \in \mathbb{N})(Q(k))$ "

Also, $Q(0) \equiv P(0)$, and $(\forall k \in \mathbb{N})(Q(k)) \equiv (\forall k \in \mathbb{N})(P(k))$

$(\forall k \in \mathbb{N})(Q(k) \implies Q(k+1))$
 $\equiv (\forall k \in \mathbb{N})((P(0) \cdots P(k)) \implies (P(0) \cdots P(k) \wedge P(k+1)))$
 $\equiv (\forall k \in \mathbb{N})((P(0) \cdots P(k)) \implies P(k+1))$

Strong Induction Principle: If $P(0)$ and

$(\forall k \in \mathbb{N})((P(0) \wedge \dots \wedge P(k)) \implies P(k+1))$,

then $(\forall k \in \mathbb{N})(P(k))$.

13/25

Well Ordering Principle and Induction.

If $(\forall n)P(n)$ is not true, then $(\exists n)\neg P(n)$.

Consider smallest m , with $\neg P(m)$, $m \geq 0$

$P(m-1) \implies P(m)$ must be false (assuming $P(0)$ holds.)

This is a proof of the induction principle!

I.e.,

$$(\neg \forall n)P(n) \implies ((\exists n)\neg(P(n-1) \implies P(n))).$$

(Contrapositive of Induction principle (assuming $P(0)$))

It assumes that there is a smallest m where $P(m)$ does not hold.

The **Well ordering principle** states that for any subset of the natural numbers there is a smallest element.

Smallest may not be what you expect: the well ordering principal holds for rationals but with different ordering!!

E.g. Reduced form is "smallest" representation of a rational number a/b .

14/25

Well ordering principle.

Thm: All natural numbers are interesting.

0 is interesting...

Let n be the first uninteresting number.

But $n-1$ is interesting and n is uninteresting,
so this is the first uninteresting number.

But this is interesting.

Thus, there is no smallest uninteresting natural number.

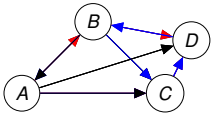
Thus: All natural numbers are interesting.

15/25

Tournaments have short cycles

Def: A round robin tournament on n players: every player p plays every other player q , and either $p \rightarrow q$ (p beats q) or $q \rightarrow p$ (q beats p .)

Def: A cycle: a sequence of $p_1, \dots, p_k, p_i \rightarrow p_{i+1}$ and $p_k \rightarrow p_1$.



Theorem: Any tournament that has a cycle has a cycle of length 3.

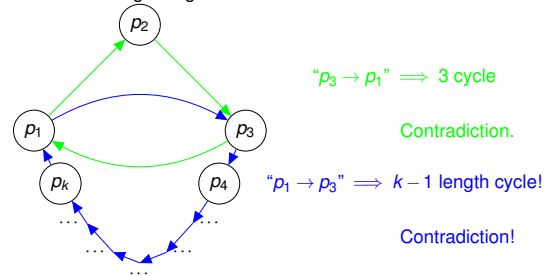
16/25

Tournament has a cycle of length 3 if at all.

Assume the the **smallest cycle** is of length k .

Case 1: Of length 3. **Done.**

Case 2: Of length larger than 3.



17/25

Tournaments have long paths.

Def: A round robin tournament on n players: every player p plays every other player q , and either $p \rightarrow q$ (p beats q) or $q \rightarrow p$ (q beats p .)

Def: A Hamiltonian path: a sequence

$$p_1, \dots, p_n, (\forall i, 0 \leq i < n) p_i \rightarrow p_{i+1}.$$

Base: True for two vertices.

(Also for one, but two is more useful as base case!)

Tournament on $n+1$ people,

Remove arbitrary person \rightarrow yield tournament on $n-1$ people.

(Result specified for each remaining pair from original tournament.)

By induction hypothesis: There is a sequence p_1, \dots, p_n contains all the people

where $p_i \rightarrow p_{i+1}$

If p is big winner, put at beginning.

If not, find first place i , where p beats p_i .

$p_1, \dots, p_{i-1}, p, p_i, \dots, p_n$ is hamiltonion path.

If no place, place at the end.

□ 18/25

Horses of the same color...

Theorem: All horses have the same color.

Base Case: $P(1)$ - trivially true.

New Base Case: $P(2)$: there are two horses with same color.

Induction Hypothesis: $P(k)$ - Any k horses have the same color.

Induction step $P(k+1)$?

First k have same color by $P(k)$. 1, 2, 2, 3, ..., $k, k+1$

Second k have same color by $P(k)$. 1, 2, 2, 3, ..., $k, k+1$

A horse in the middle in common! 1, 2, 2, 3, ..., $k, k+1$

All k must have the same color! 1, 2, 3, ..., $k, k+1$

How about $P(1) \implies P(2)$?

Fix base case.

...Still doesn't work!!

(There are two horses is \neq For all two horses!!!)

Of course it doesn't work.

As we will see, it is more subtle to catch errors in proofs of correct theorems!!

19/25

Strong Induction and Recursion.

Thm: For every natural number $n \geq 12$, $n = 4x + 5y$.

Instead of proof, let's write some code!

```
def find-x-y(n):
    if (n==12) return (3,0)
    elif (n==13): return (2,1)
    elif (n==14): return (1,2)
    elif (n==15): return (0,3)
    else:
        (x',y') = find-x-y(n-4)
        return (x'+1,y')
```

Base cases: $P(12)$, $P(13)$, $P(14)$, $P(15)$. Yes.

Strong Induction step:

Recursive call is correct: $P(n-4) \implies P(n)$.

$n-4 = 4x' + 5y' \implies n = 4(x'+1) + 5(y')$

Slight differences: showed for all $n \geq 16$ that $\bigwedge_{i=4}^{n-1} P(i) \implies P(n)$.

20/25

Sad Islanders...

Island with 100 possibly blue-eyed and green-eyed inhabitants.

Any islander who knows they have green eyes must kill themselves that day.

No islander knows their own eye color, but knows everyone else's.

All islanders have green eyes!

First rule of island: Don't talk about eye color!

Visitor: "I see someone has green eyes."

Result: On day 100, they all kill themselves.

Why?

21/25

They know induction.

Thm: If there are n villagers with green eyes they kill themselves on day n .

Proof:

Base: $n = 1$. Person with green eyes kills themselves on day 1.

Induction hypothesis:

If there were n people with green eyes, they would pass away on day n .

Induction step:

On day $n+1$, a green-eyed person sees n people with green eyes.

But they didn't kill themselves.

So there must be $n+1$ people with green eyes.

One of them, is me.

Sad. □

Wait! Visitor added no information.

22/25

Common Knowledge.

Using knowledge about what other people's knowledge (your eye color) is.

On day 1, everyone knows everyone sees more than zero.

On day 2, everyone knows everyone sees more than one.

...

On day 99, no one sees 98 since everyone knows everyone else does not see 97...

On day 100, ...uh oh!

Another example:

Emperor's new clothes!

No one knows other people see that he has no clothes.

Until kid points it out.

23/25

Summary: principle of induction.

Today: More induction.

$(P(0) \wedge (\forall k \in \mathbb{N})(P(k) \implies P(k+1))) \implies (\forall n \in \mathbb{N})(P(n))$

Statement to prove: $P(n)$ for n starting from n_0

Base Case: Prove $P(n_0)$.

Ind. Step: Prove. For all values, $n \geq n_0$, $P(n) \implies P(n+1)$.

Statement is proven!

Strong Induction:

$(P(0) \wedge (\forall n \in \mathbb{N})(P(n) \implies P(n+1))) \implies (\forall n \in \mathbb{N})(P(n))$

Also Today: strengthened induction hypothesis.

Strengthen theorem statement.

Sum of first n odds is n^2 .

Hole anywhere.

Not same as strong induction. E.g., used in product of primes proof.

Induction \equiv Recursion.

24/25

Summary: principle of induction.

$$(P(0) \wedge ((\forall k \in \mathbb{N})(P(k) \implies P(k+1)))) \implies (\forall n \in \mathbb{N})(P(n))$$

Variations:

$$(P(0) \wedge ((\forall n \in \mathbb{N})(P(n) \implies P(n+1)))) \implies (\forall n \in \mathbb{N})(P(n))$$

$$(P(1) \wedge ((\forall n \in \mathbb{N})((n \geq 1) \wedge P(n) \implies P(n+1)))) \\ \implies (\forall n \in \mathbb{N})((n \geq 1) \implies P(n))$$

Statement to prove: $P(n)$ for n starting from n_0

Base Case: Prove $P(n_0)$.

Ind. Step: Prove. For all values, $n \geq n_0$, $P(n) \implies P(n+1)$.

Statement is proven!