Review

Now...

Induction

$$P(0) \wedge ((\forall n)(P(n) \implies P(n+1) \equiv (\forall n \in N)\ P(n).$$

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Thm: For all $n \geq 1$, $8|3^{2n} - 1$.

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$3^{2n+2} - 1 = 9(3^{2n}) - 1$

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Base: $8 | 3^2 - 1$.

Induction Hypothesis: Assume $P(n)$: True for some n.
$(3^{2n} - 1 = 8d)$

Induction Step: Prove $P(n+1)$

$3^{2n+2} - 1 = 9(3^{2n}) - 1$ (by induction hypothesis)

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$\square$

Stable Marriage: a study in definitions and WOP.

n-men, n-women.

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No, for roommates problem.

TMA.

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Not rogue couple!

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Connected Graph: one connected component.

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Thm: Every connected graph where every vertex has even degree has an Eulerian Tour; a tour which visits every edge exactly once.

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Given $G = (V, E)$, a coloring of a G assigns colors to vertices V where for each edge the endpoints have different colors.

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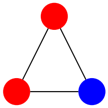

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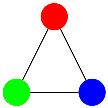
Given $G = (V, E)$, a coloring of a G assigns colors to vertices V where for each edge the endpoints have different colors.



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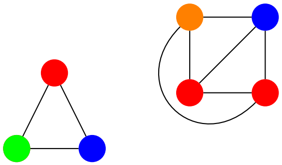

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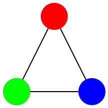

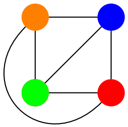

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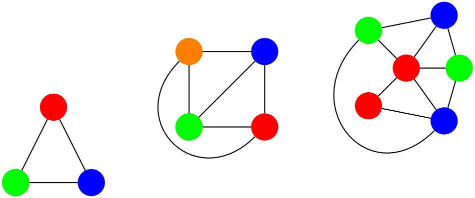

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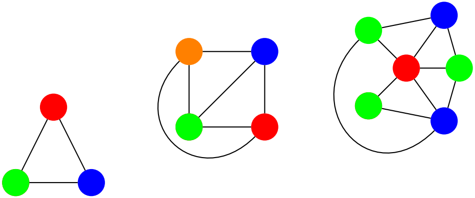

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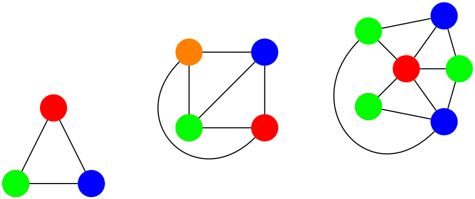

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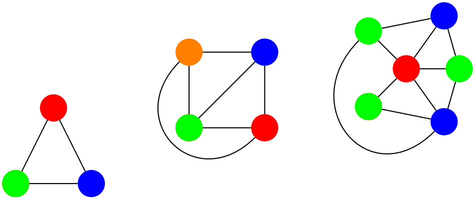

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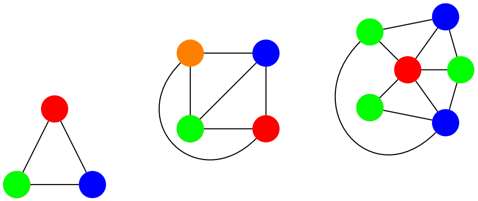

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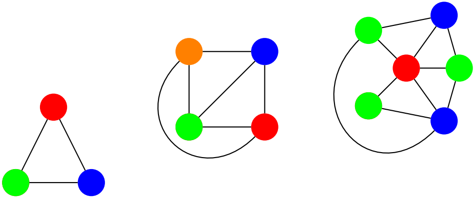

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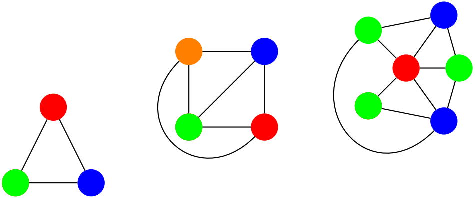

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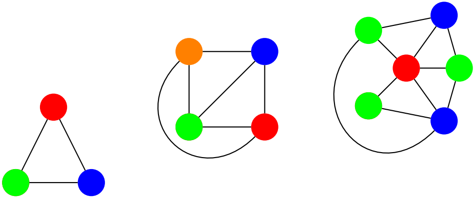

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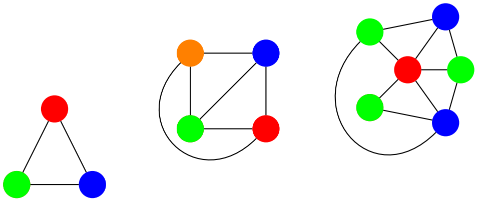

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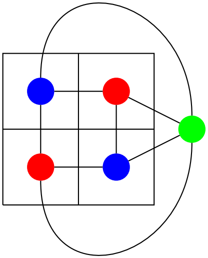
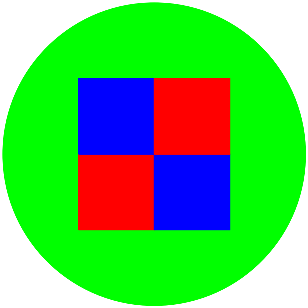
Interesting things to do. Algorithm!

Planar graphs and maps.

Planar graph coloring $\equiv$ map coloring.





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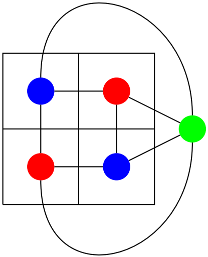

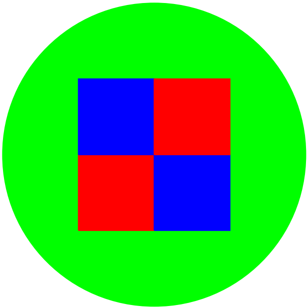

Four color theorem is about planar graphs!

Six color theorem.

Theorem: Every planar graph can be colored with six colors.

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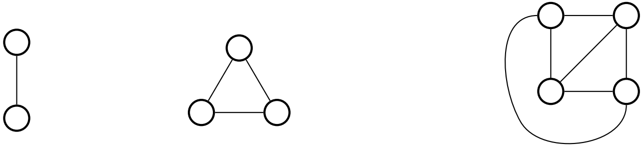

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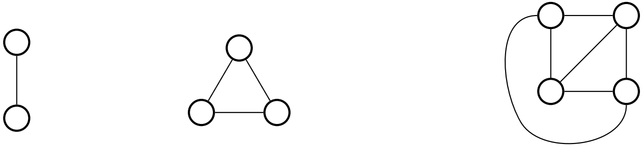

K_n, $|V| = n$

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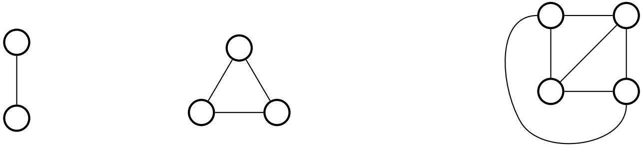

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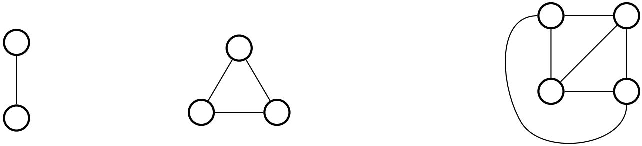

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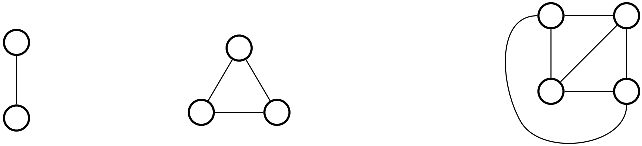

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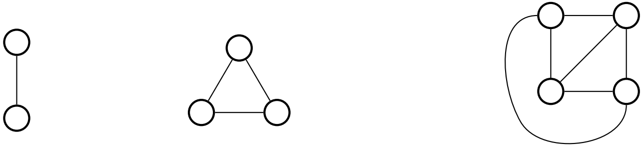

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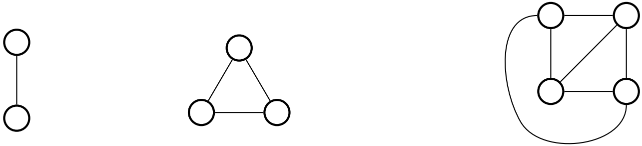

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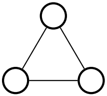

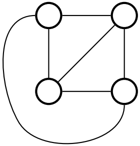

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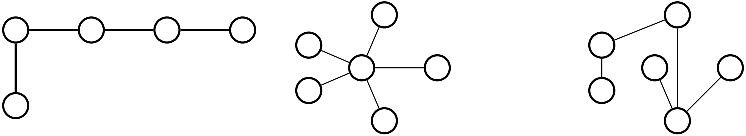
Trees.



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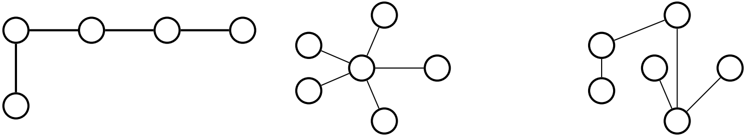

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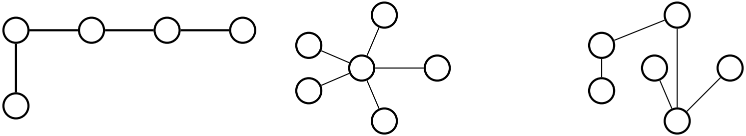

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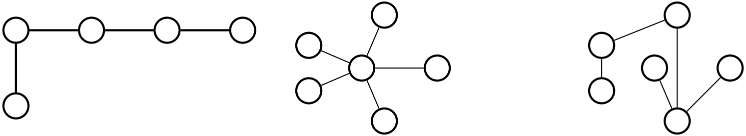

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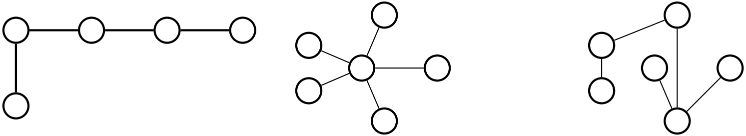

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An acyclic graph where any edge addition creates a cycle.

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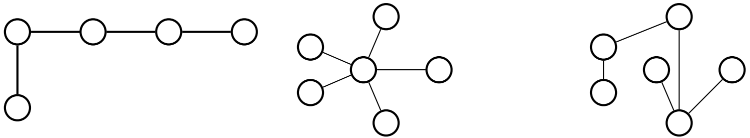

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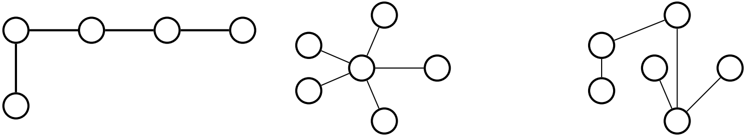

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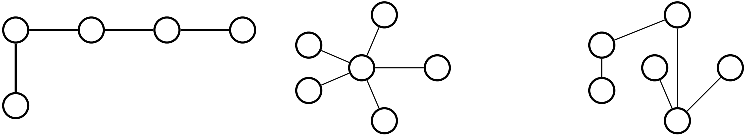

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Hypercube

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Hypercubes. Really connected.

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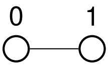

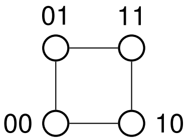

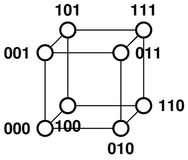

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Arithmetic modulo *m*.

Elements of equivalence classes of integers.

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Additive inverses are intuitively negative numbers.

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$d = e^{-1} = -17 = 43 = \pmod{60}$

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Example: How many permutations of n items start with 1 or 2?
$1 \times (n-1)! + 1 \times (n-1)!$

Inclusion/Exclusion Rule: For any S **and** T**,**
$|S \cup T| = |S| + |T| - |S \cap T|$.

Example: How many 10-digit phone numbers have 7 as their first or second digit?

S = phone numbers with 7 as first digit.$|S| = 10^9$

T = phone numbers with 7 as second digit. $|T| = 10^9$.

$S \cap T$ = phone numbers with 7 as first and second digit.

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Answer: $|S| + |T| - |S \cap T| = 10^9 + 10^9 - 10^8$.

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Concept 1: can call program A
Concept 2:One can modify text of input program (to HALT).

CS70: Review of Probability.

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Match Items

[1] $Pr[X \geq a] \leq \frac{E[f(X)]}{f(a)}$;

[2] $Pr[|X - E[X]| > a] \leq \frac{var[X]}{a^2}$

[3] $Pr[X \geq a] \leq \min_{\theta > 0} \frac{E[e^{\theta X}]}{e^{\theta a}}$

[4] $g(\cdot)$ convex $\Rightarrow E[g(X)] \geq g(E[X])$

[5] $E[Y] + \frac{cov(X,Y)}{var(X)}(X - E[X])$.

[6] $\sum_y y Pr[Y = y | X = x]$

[7] $Pr[|\frac{X_1 + \cdots + X_n}{n} - E[X_1]| \geq \varepsilon] \rightarrow 0$,

[8] $E[(Y - E[Y|X])h(X)] = 0$.

Match Items

[1] $Pr[X \geq a] \leq \frac{E[f(X)]}{f(a)}$;

[2] $Pr[|X - E[X]| > a] \leq \frac{var[X]}{a^2}$

[3] $Pr[X \geq a] \leq \min_{\theta > 0} \frac{E[e^{\theta X}]}{e^{\theta a}}$

[4] $g(\cdot)$ convex $\Rightarrow E[g(X)] \geq g(E[X])$

[5] $E[Y] + \frac{cov(X,Y)}{var(X)}(X - E[X])$.

[6] $\sum_y y Pr[Y = y | X = x]$

[7] $Pr[|\frac{X_1 + \cdots + X_n}{n} - E[X_1]| \geq \epsilon] \rightarrow 0$,

[8] $E[(Y - E[Y|X])h(X)] = 0$.

- WLLN

Match Items

[1] $Pr[X \geq a] \leq \frac{E[f(X)]}{f(a)}$

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[8] $E[(Y - E[Y|X])h(X)] = 0$.

- WLLN (7)

Match Items

[1] $Pr[X \geq a] \leq \frac{E[f(X)]}{f(a)}$;

[2] $Pr[|X - E[X]| > a] \leq \frac{var[X]}{a^2}$

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[8] $E[(Y - E[Y|X])h(X)] = 0$.

- WLLN (7)
- MMSE

Match Items

[1] $Pr[X \geq a] \leq \frac{E[f(X)]}{f(a)}$

[2] $Pr[|X - E[X]| > a] \leq \frac{var[X]}{a^2}$

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- WLLN (7)
- MMSE (6)

Match Items

[1] $Pr[X \geq a] \leq \frac{E[f(X)]}{f(a)}$;

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- WLLN (7)
- MMSE (6)
- Projection property

Match Items

[1] $Pr[X \geq a] \leq \frac{E[f(X)]}{f(a)}$

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- WLLN (7)
- MMSE (6)
- Projection property (8)

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- WLLN (7)
- MMSE (6)
- Projection property (8)
- Chebyshev

Match Items

[1] $Pr[X \geq a] \leq \frac{E[f(X)]}{f(a)}$

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- WLLN (7)
- MMSE (6)
- Projection property (8)
- Chebyshev (2)

Match Items

[1] $Pr[X \geq a] \leq \frac{E[f(X)]}{f(a)}$

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- WLLN (7)
- MMSE (6)
- Projection property (8)
- Chebyshev (2)
- LLSE

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- WLLN (7)
- MMSE (6)
- Projection property (8)
- Chebyshev (2)
- LLSE (5)

Match Items

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- WLLN (7)
- MMSE (6)
- Projection property (8)
- Chebyshev (2)
- LLSE (5)
- Markov's inequality

Match Items

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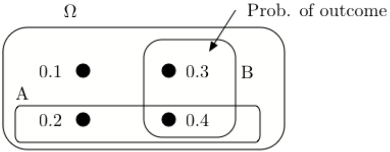
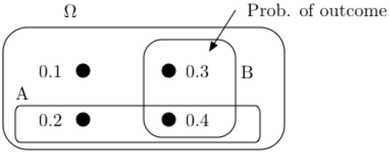
- WLLN (7)
- MMSE (6)
- Projection property (8)
- Chebyshev (2)
- LLSE (5)
- Markov's inequality (1)

Quiz 1: G

Quiz 1: G

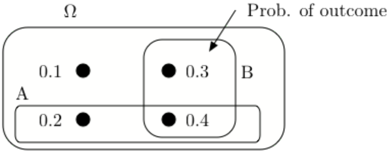


Quiz 1: G



1. What is $P[A|B]$?

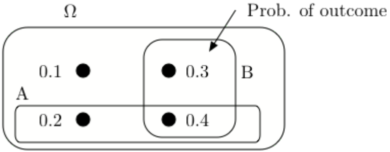
Quiz 1: G



1. What is $P[A|B]$?

   $Pr[A|B] =$

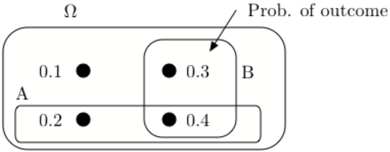
Quiz 1: G



1. What is $P[A|B]$?

   $Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]} =$

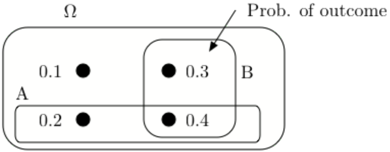
Quiz 1: G



1. What is $P[A|B]$?

   $Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]} = \frac{0.4}{0.7}$

Quiz 1: G



1. What is $P[A|B]$?

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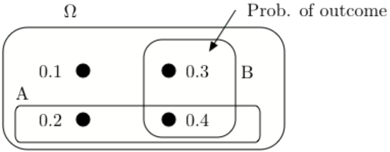
Quiz 1: G



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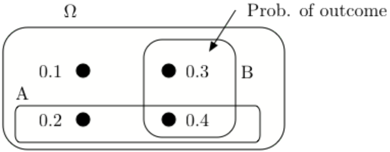
Quiz 1: G



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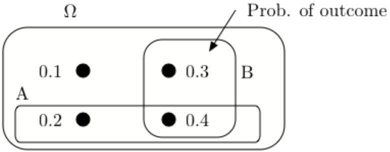

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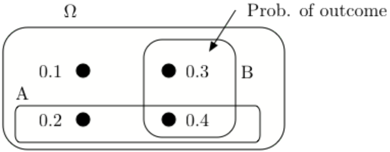
Quiz 1: G



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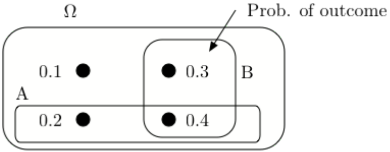
   $$Pr[B|A] = \frac{Pr[A \cap B]}{Pr[A]} = \frac{0.4}{0.6}$$

3. Are A and B positively correlated?

Quiz 1: G



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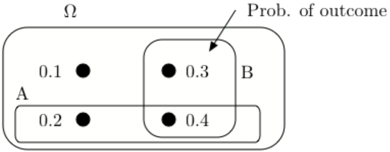
3. Are A and B positively correlated?

   No.

Quiz 1: G



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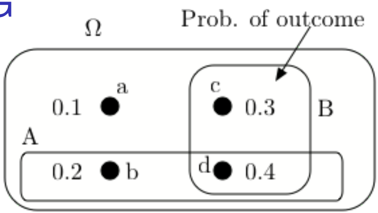
3. Are A and B positively correlated?

   No. $Pr[A \cap B] = 0.4 < Pr[A]Pr[B] = 0.6 \times 0.7$.

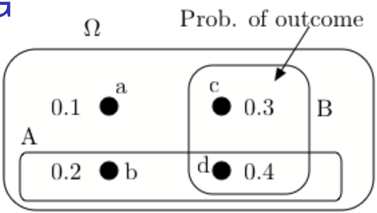
Quiz 1: G

Quiz 1: G



ω	$X(\omega)$	$Y(\omega)$
a	0	0
b	1	0
c	0	2
d	1	2

Quiz 1: G



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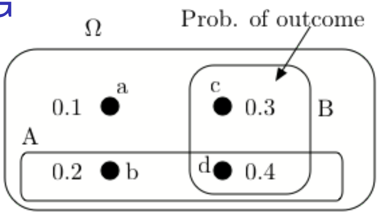
4. What is $E[Y|X]$?

Quiz 1: G



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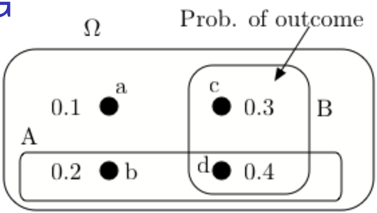
4. What is $E[Y|X]$?

$$E[Y|X=0] \quad =$$

Quiz 1: G



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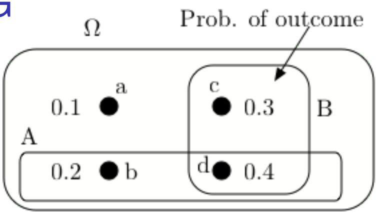
$$E[Y|X=0] = 0 \times Pr[Y=0|X=0] + 2 \times Pr[Y=2|X=0]$$

Quiz 1: G



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$$=$$

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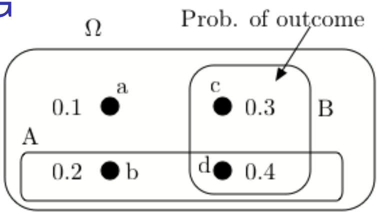

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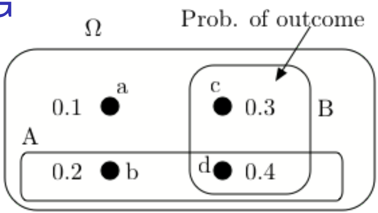
$$\begin{aligned} E[Y|X=0] &= 0 \times Pr[Y=0|X=0] + 2 \times Pr[Y=2|X=0] \\ &= 2 \times \frac{0.3}{0.4} = \end{aligned}$$

Quiz 1: G



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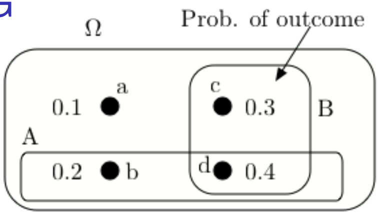
$$

Quiz 1: G



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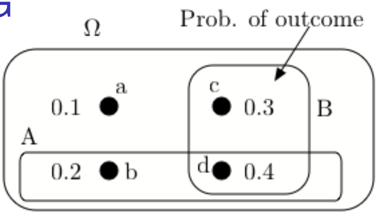
Quiz 1: G



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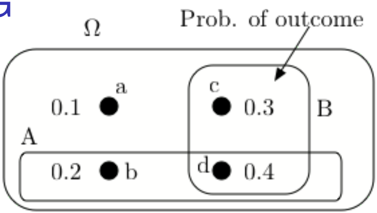

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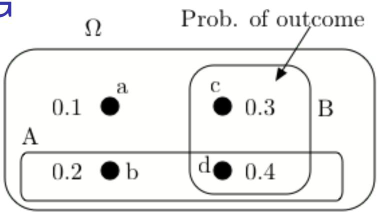

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&= 2 \times \frac{0.4}{0.6} =
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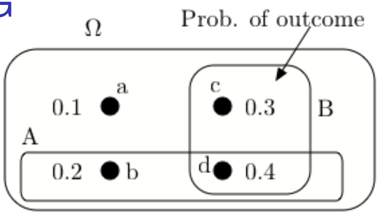

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c	0	2
d	1	2

4. What is $E[Y|X]$?

$$\begin{aligned}
E[Y|X=0] &= 0 \times Pr[Y=0|X=0] + 2 \times Pr[Y=2|X=0] \\
&= 2 \times \frac{0.3}{0.4} = 1.5 \\
E[Y|X=1] &= 0 \times Pr[Y=0|X=1] + 2 \times Pr[Y=2|X=1] \\
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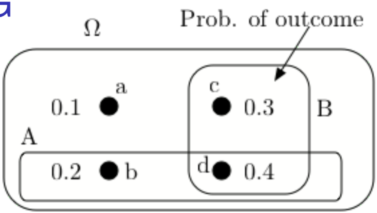
\end{aligned}$$

Quiz 1: G



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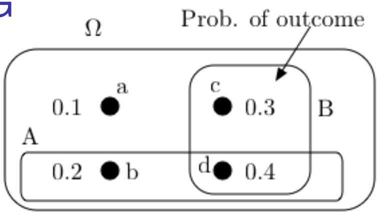
5. What is $cov(X, Y)$?

Quiz 1: G



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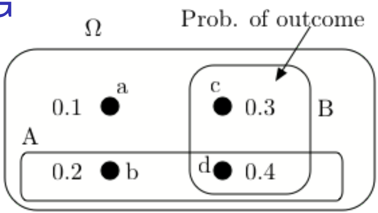
Quiz 1: G



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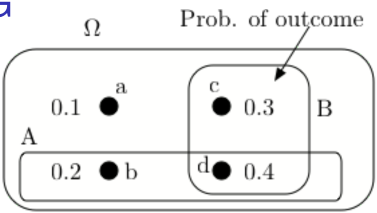
Quiz 1: G



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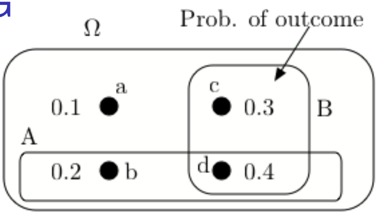
$cov(X, Y) = E[XY] - E[X]E[Y] = 0.8 - 0.6 \times 1.4 =$

Quiz 1: G



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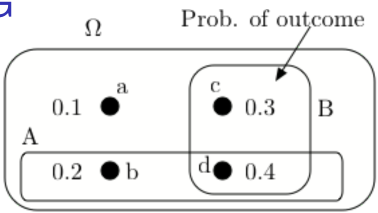
5. What is $cov(X, Y)$?

$cov(X,Y) = E[XY] - E[X]E[Y] = 0.8 - 0.6 \times 1.4 = -0.04$

Quiz 1: G



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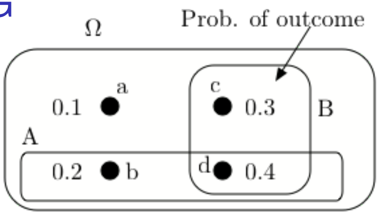
6. What is $L[Y|X]$?

Quiz 1: G



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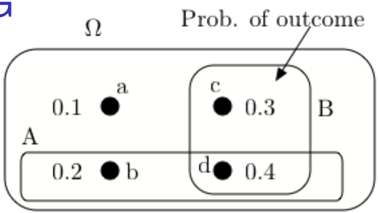
Quiz 1: G



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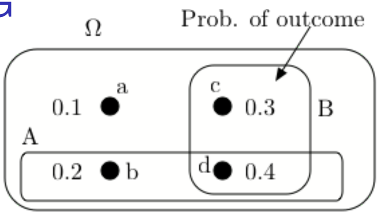
   $L[Y|X] = E[Y] + \frac{cov(X,Y)}{var(X)}(X - E[X]) =$

Quiz 1: G



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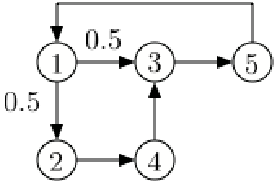
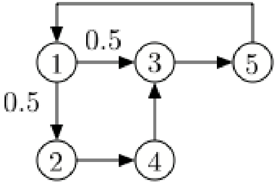
   $L[Y|X] = E[Y] + \frac{cov(X,Y)}{var(X)}(X - E[X]) = 1.4 + \frac{-0.04}{0.6 \times 0.4}(X - 0.6)$

Quiz 1: G

Quiz 1: G

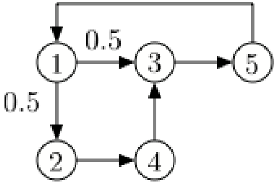


Quiz 1: G



7. Is this Markov chains irreducible?

Quiz 1: G



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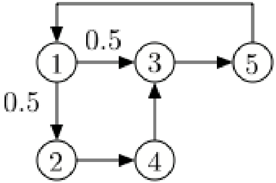
Quiz 1: G



7. Is this Markov chains irreducible? Yes.
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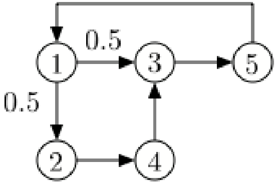
Quiz 1: G



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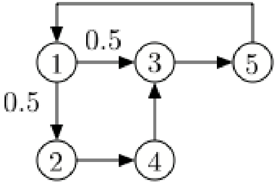
   No.

Quiz 1: G



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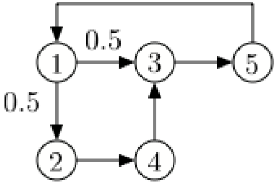
Quiz 1: G



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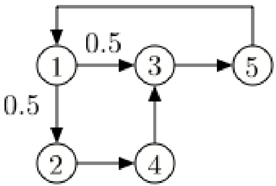
   No. The return times to 3 are $\{3, 5, ..\}$:

Quiz 1: G



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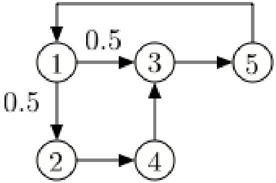
   No. The return times to 3 are $\{3, 5, ..\}$: coprime!

Quiz 1: G



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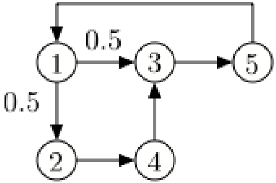
Quiz 1: G



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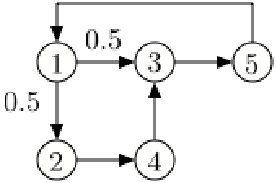

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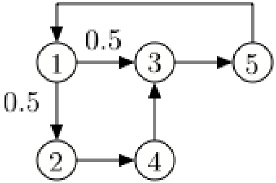
10. Does $\frac{1}{n}\sum_{m=1}^{n-1} 1\{X_m = 3\}$ converge as $n \to \infty$?

Quiz 1: G



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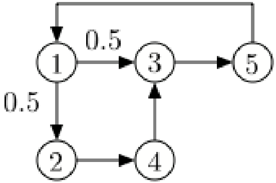

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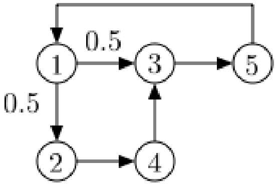
Quiz 1: G



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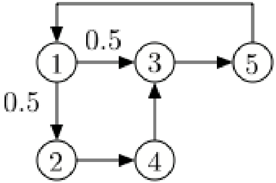
11. Calculate π.

    Let $a = \pi(1)$.

Quiz 1: G



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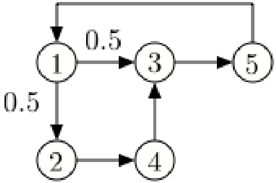
11. Calculate π.

    Let $a = \pi(1)$. Then $a = \pi(5)$,

Quiz 1: G



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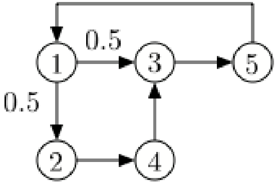
    Let $a = \pi(1)$. Then $a = \pi(5), \pi(2) = 0.5a$,

Quiz 1: G



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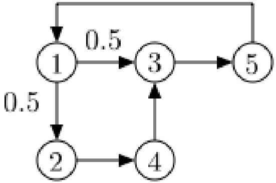
    Let $a = \pi(1)$. Then $a = \pi(5), \pi(2) = 0.5a, \pi(4) = \pi(2) = 0.5a,$

Quiz 1: G



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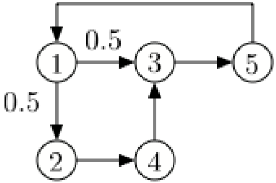
Quiz 1: G



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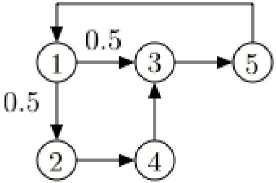
Quiz 1: G



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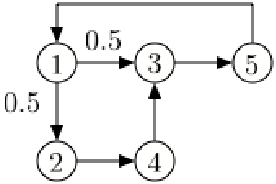
    Let $a = \pi(1)$. Then $a = \pi(5), \pi(2) = 0.5a, \pi(4) = \pi(2) = 0.5a, \pi(3) = 0.5\pi(1) + \pi(4) = a$. Thus,
    $\pi = [a, 0.5a, a, 0.5a, a] = [1, 0.5, 1, 0.5, 1]a$, so $a = 1/4$.

Quiz 1: G

Quiz 1: G



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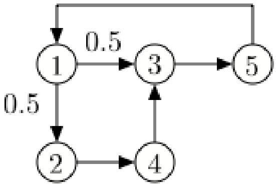
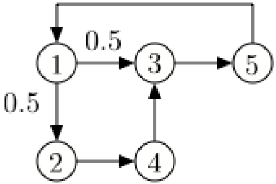


12. Write the first step equations for calculating the mean time from 1 to 4.

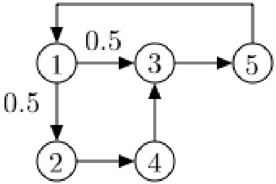
Quiz 1: G



12. Write the first step equations for calculating the mean time from 1 to 4.

$$\beta(1) = 1 + 0.5\beta(2) + 0.5\beta(3)$$

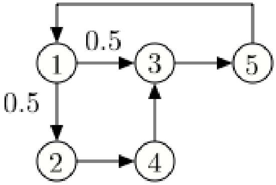
Quiz 1: G



12. Write the first step equations for calculating the mean time from 1 to 4.

$$\beta(1) = 1 + 0.5\beta(2) + 0.5\beta(3)$$
$$\beta(2) = 1$$

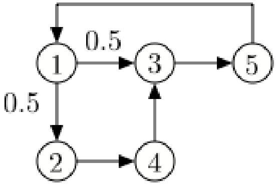
Quiz 1: G



12. Write the first step equations for calculating the mean time from 1 to 4.

$$\beta(1) = 1 + 0.5\beta(2) + 0.5\beta(3)$$
$$\beta(2) = 1$$
$$\beta(3) = 1 + \beta(5)$$

Quiz 1: G



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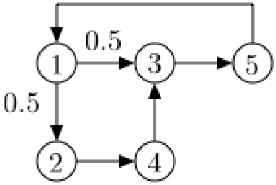

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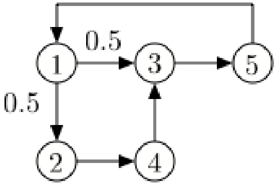
Quiz 1: G



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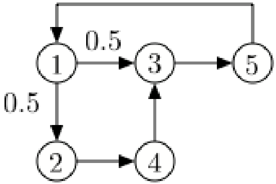
13. Solve these equations.

$$\beta(1) \quad = \quad 1 + 0.5 \times 1 + 0.5 \times (1 + (1 + \beta(1)))$$

Quiz 1: G



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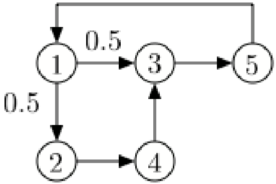
$$\begin{aligned}\beta(1) &= 1 + 0.5 \times 1 + 0.5 \times (1 + (1 + \beta(1))) \\ &= 2.5 + 0.5\beta(1).\end{aligned}$$

Quiz 1: G



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$$\begin{aligned} \beta(1) &= 1 + 0.5 \times 1 + 0.5 \times (1 + (1 + \beta(1))) \\ &= 2.5 + 0.5\beta(1). \end{aligned}$$

Hence, $\beta(1) = 5$.

Quiz 1: G

14. Which is $E[Y|X]$? Blue, red or green?

Quiz 1: G

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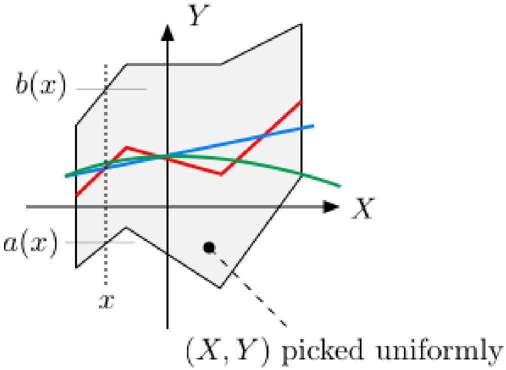

Quiz 1: G

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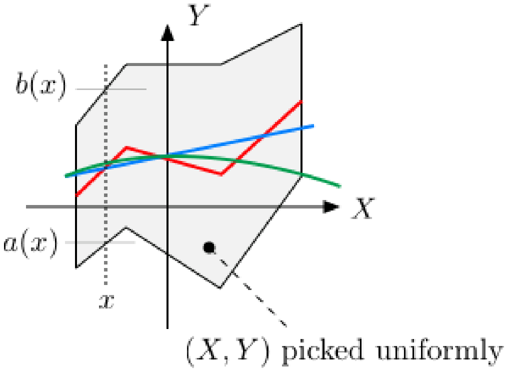
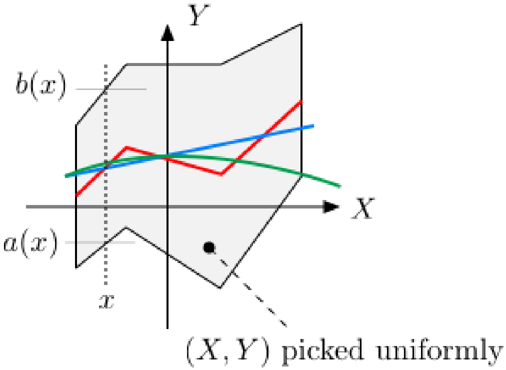


Answer: Red.

Quiz 1: G

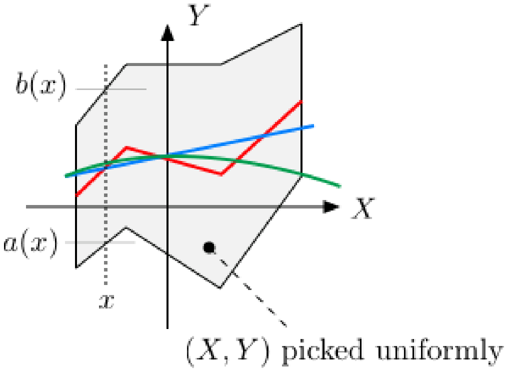
14. Which is $E[Y|X]$? Blue, red or green?



Answer: Red.
Given $X = x, Y = U[a(x), b(x)]$.

Quiz 1: G

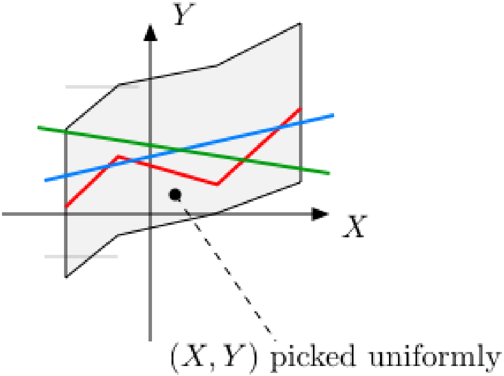
14. Which is $E[Y|X]$? Blue, red or green?



Answer: Red.
Given $X = x, Y = U[a(x), b(x)]$. Thus, $E[Y|X = x] = \frac{a(x)+b(x)}{2}$.

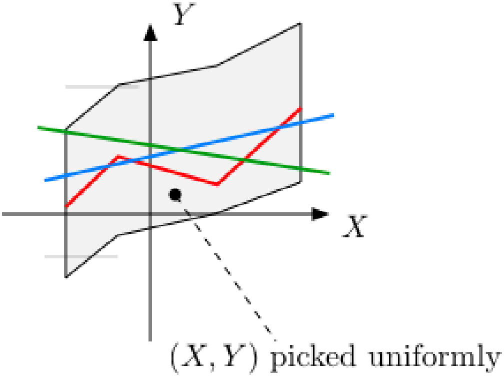
Quiz 1: G

15. Which is $L[Y|X]$? Blue, red or green?



Quiz 1: G

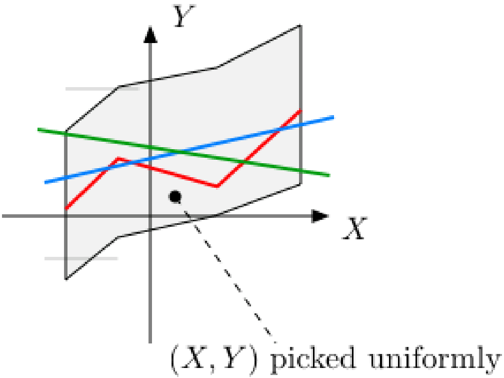
15. Which is $L[Y|X]$? Blue, red or green?



Answer: Blue.

Quiz 1: G

15. Which is $L[Y|X]$? Blue, red or green?



Answer: Blue.
Cannot be red (not a straight line).

Quiz 1: G

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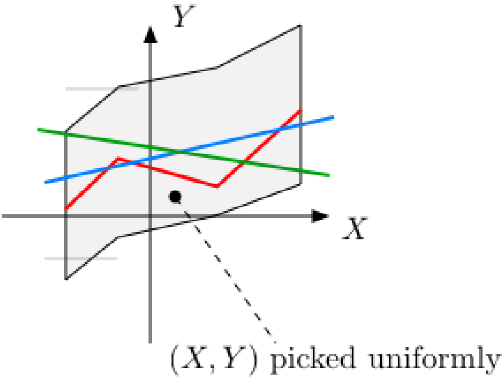

Answer: Blue.
Cannot be red (not a straight line).
Cannot be green: X and Y are clearly positively correlated.

Quiz 2: PG

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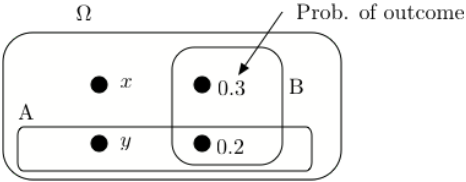

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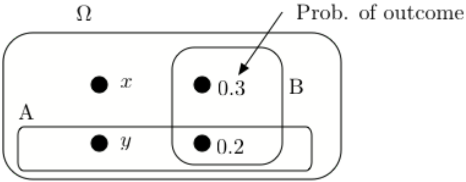
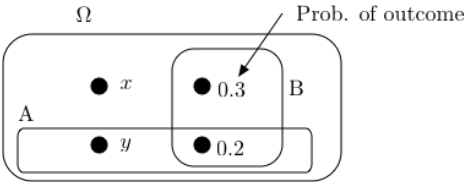


1. Find (x, y) so that A and B are independent.

Quiz 2: PG



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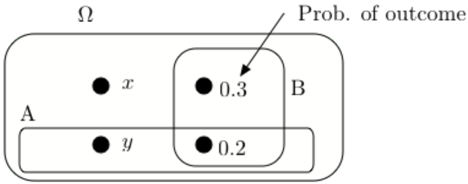
   We need
   $$Pr[A \cap B] = Pr[A]Pr[B]$$

Quiz 2: PG



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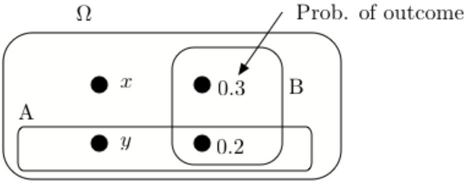
   $$0.2 = (y + 0.2) \times 0.5 =$$

Quiz 2: PG



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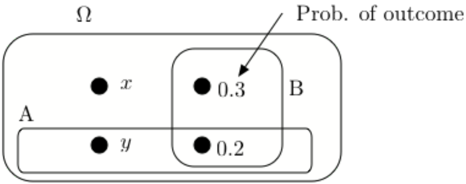

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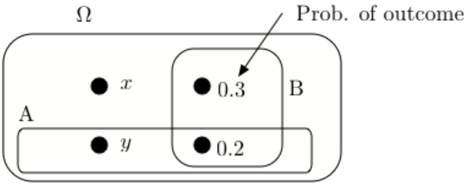
Quiz 2: PG



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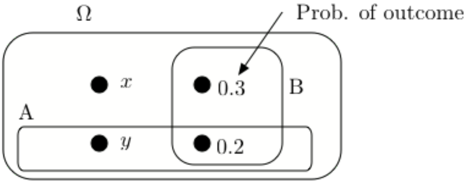
Quiz 2: PG



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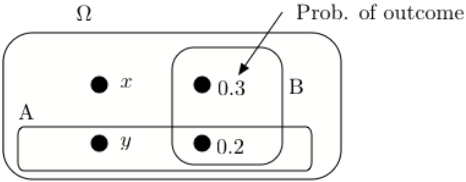
Quiz 2: PG



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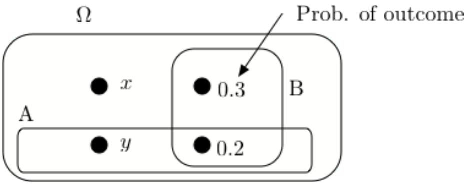
Quiz 2: PG



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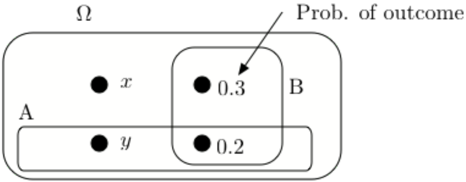

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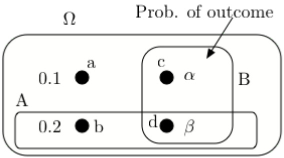
2. Find the value of x that maximizes $Pr[B|A]$.

   Obviously, it is $x = 0.5$.

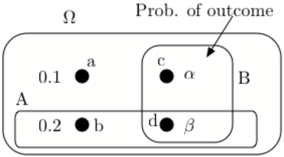
Quiz 2: PG

Quiz 2: PG



ω	$X(\omega)$	$Y(\omega)$
a	0	0
b	1	0
c	0	2
d	1	2

Quiz 2: PG



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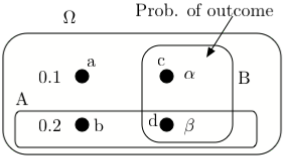
3. Find α so that X and Y are independent.

Quiz 2: PG



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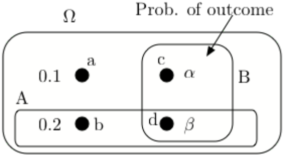
$$Pr[X = 0, Y = 0] = Pr[X = 0]Pr[Y = 0]$$

Quiz 2: PG



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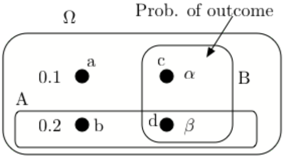
$$0.1 = (0.1+\alpha)\times(0.1+0.2) =$$

Quiz 2: PG



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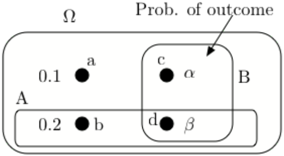
Quiz 2: PG



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Hence,

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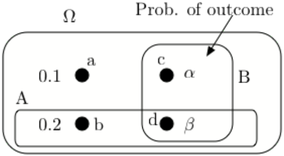
Quiz 2: PG



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Quiz 2: PG

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4. A CS70 student is great w.p. 0.3 and good w.p. 0.7.

Quiz 2: PG

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Quiz 2: PG

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Quiz 2: PG

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Quiz 2: PG

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Quiz 2: PG

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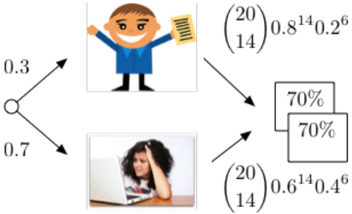

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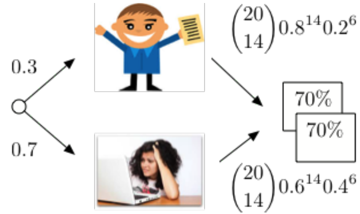

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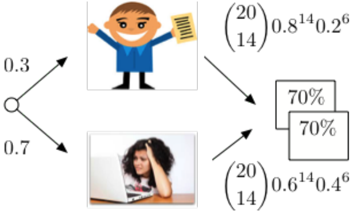

$$p := Pr[\text{great}|\text{scores}] = \frac{0.3\binom{20}{14}0.8^{14}0.2^{6}}{0.3\binom{20}{14}0.8^{14}0.2^{6} + 0.7\binom{20}{14}0.6^{14}0.4^{6}}$$

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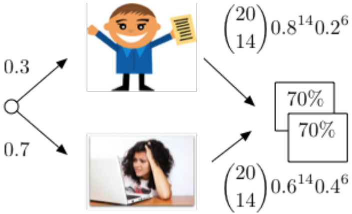

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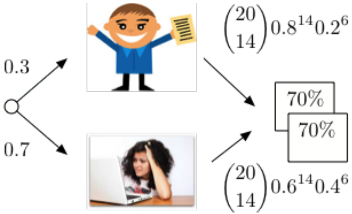

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$$= \frac{(0.3)0.8^{14}0.2^6}{(0.3)0.8^{14}0.2^6 + (0.7)0.6^{14}0.4^6} \approx 0.27$$

Quiz 2: PG

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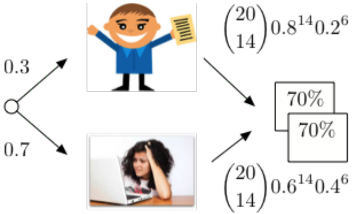

$$p := Pr[\text{great}|\text{scores}] = \frac{0.3\binom{20}{14}0.8^{14}0.2^{6}}{0.3\binom{20}{14}0.8^{14}0.2^{6} + 0.7\binom{20}{14}0.6^{14}0.4^{6}}$$

$$= \frac{(0.3)0.8^{14}0.2^{6}}{(0.3)0.8^{14}0.2^{6} + (0.7)0.6^{14}0.4^{6}} \approx 0.27$$

Expected score $=$

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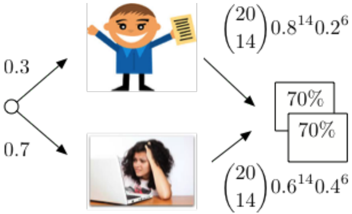

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Expected score $= p80\% + (1-p)60\% \approx$

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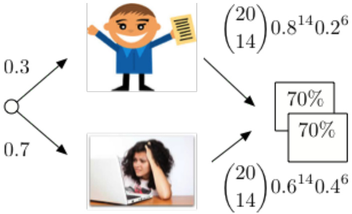

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Expected score $= p80\% + (1-p)60\% \approx 65\%.$

Quiz 2: PG

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Quiz 2: PG

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Quiz 2: PG

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Hence,

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Q2: PG

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$$E[X_{n+1}|X_n] \quad = \quad X_n \times [10\frac{1}{6} + 0.5 \times \frac{5}{6}]$$

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$$\begin{aligned} E[X_{n+1}|X_n] &= X_n \times [10\frac{1}{6} + 0.5 \times \frac{5}{6}] \\ &= \rho X_n, \rho = 10\frac{1}{6} + 0.5 \times \frac{5}{6} \end{aligned}$$

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Thus,

$$E[X_n] = \rho^{n-1}, n \geq 1.$$

Quiz 3: R

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   Hint: If $X = Expo(\lambda), f_X(x) = \lambda e^{-\lambda x} 1\{x > 0\}, E[X] = 1/\lambda$.

Quiz 3: R

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$$
(b) \qquad E[\text{ lifespan of other bulb }] = p \times 1 + (1-p) \times 0.8 \approx 0.9.
$$

Quiz 3: R

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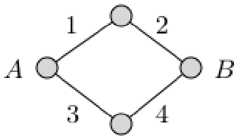

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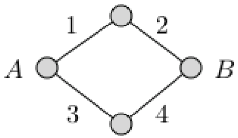

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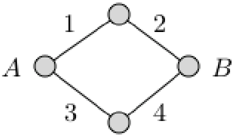
Quiz 3: R



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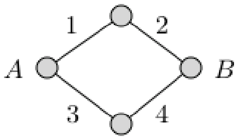

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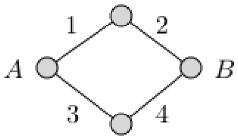
Quiz 3: R



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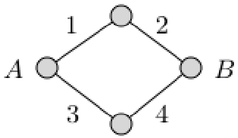

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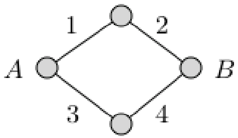

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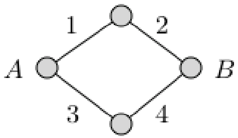
Quiz 3: R



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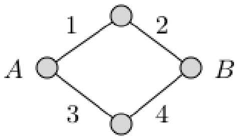
Quiz 3: R



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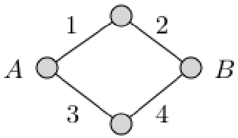
Quiz 3: R



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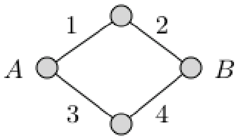
Quiz 3: R



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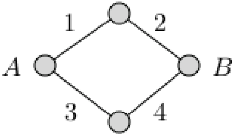
Quiz 3: R



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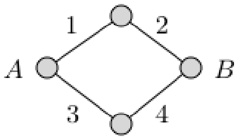
Quiz 3: R



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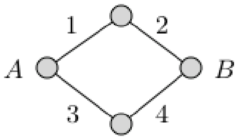
Quiz 3: R



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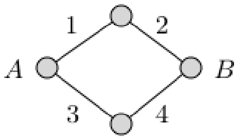
Quiz 3: R



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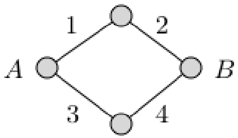
f_Z(t) &= 8t - 12t^2 + 4t^3 \\
E[Z] &= \int_0^1 t f_Z(t) dt = 8\frac{1}{3} - 12\frac{1}{4} + 4\frac{1}{5}
\end{aligned}
$$

Quiz 3: R



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&\approx 0.4667.
\end{aligned}
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Quiz 3: R

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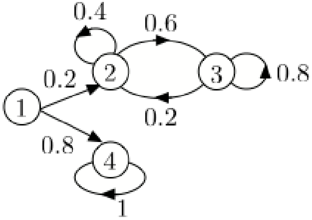
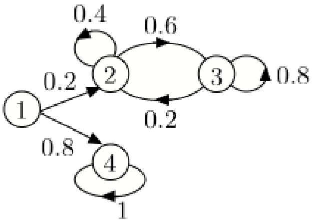
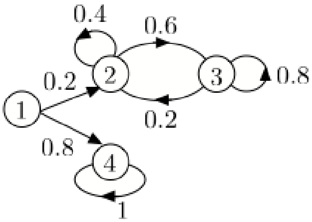


Quiz 3: R



3. We are given π_0.

Quiz 3: R



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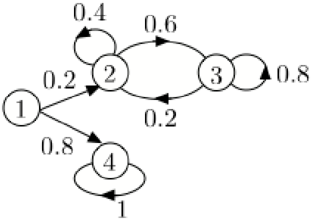
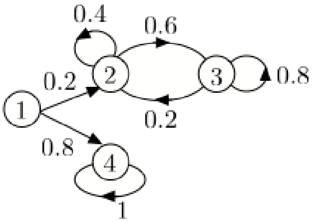
Quiz 3: R



3. We are given π_0. Find $\lim_{n\to\infty} \pi_n$.
   With probability $\alpha := 0.2\pi_0(1) + \pi_0(2) + \pi_0(3)$, the MC ends up in $\{2,3\}$.

Quiz 3: R



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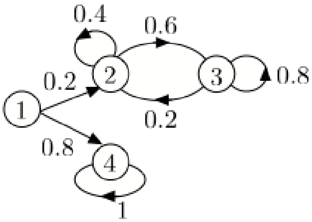

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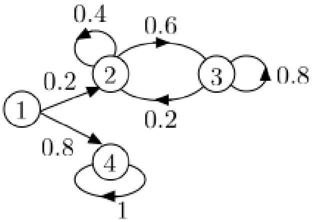
Quiz 3: R



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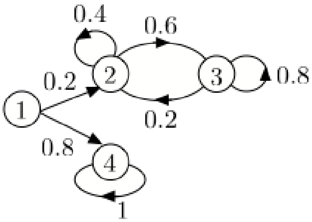

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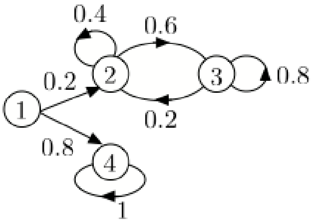

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Quiz 3: R

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- $X = B(n,p) \implies var(X) = n^2 p(1-p)$. No.
- $E[X] = E[X|A] + E[X|\bar{A}]$. No.
- $\sum_{n=0}^{\infty} a^n = 1/a$. No.
- CS70 is difficult. No.
- I will do poorly on the final. No.

Common Mistakes

- $\Omega = \{1,2,3\}$. Define X, Y with $cov(X,Y) = 0$ and X, Y not independent.

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