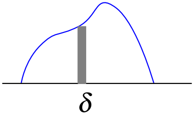
Probability

Probability!
Confuses us. But really neat.
At times, continuous. At others, discrete.

Sample Space: Ω, $Pr[\omega]$.
Event: $Pr[A] = \sum_{\omega \in A} Pr[\omega]$
$\sum_{\omega} Pr[\omega] = 1$.
Random variables: $X(\omega)$.
Distribution: $Pr[X = x]$
$\sum_{x} Pr[X = x] = 1$.

Random Variable: X
Event: $A = [a, b]$, $Pr[X \in A]$,
CDF: $F(x) = Pr[X \leq x]$.
PDF: $f(x) = \frac{dF(x)}{dx}$.
$\int_{-\infty}^{\infty} f(x) = 1$.

Continuous as Discrete.
$Pr[X \in [x, x+\delta]] \approx f(x)\delta$



Probability Rules are all good.

Conditional Probabity.

Events: A, B

Discrete: "Heads", "Tails", $X = 1$, $Y = 5$.

Continuous: X in $[.2, .3]$. $X \in [.2, .3]$ or $X \in [.4, .6]$.

Conditional Probability: $Pr[A|B] = \frac{Pr[A] \cap Pr[B]}{Pr[B]}$

$Pr[\text{"Second Heads"}|\text{"First Heads"}]$,
$Pr[X \in [.2, .3] | X \in [.2, .3] \text{ or } X \in [.5, .6]]$.

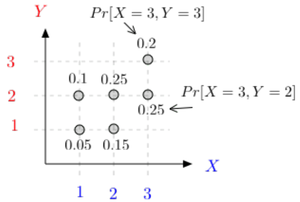
Total Probability Rule: $Pr[A] = Pr[A \cap B] + Pr[A \cap \overline{B}]$
$Pr[\text{"Second Heads"}] = Pr[HH] + Pr[HT]$
B is First coin heads.
$Pr[X \in [.45, .55]] = Pr[X \in [.45, .50]] + Pr[X \in (.50, .55]]$
B is $X \in [0, .5]$

Product Rule: $Pr[A \cap B] = Pr[A|B]Pr[B]$.
Bayes Rule: $Pr[A|B] = Pr[B|A]Pr[A]/Pr[B]$.

All work for continuous with intervals as events.

Joint distribution.

Y/X	1	2	4	8	
1	.03	.05	.1	.02	.20
2	.2	.01	.03	.02	.26
3	.21	.06	.03	.01	.31
5	.02	.2	.02	.01	.25
	.44	.32	.18	.06	



Marginal Distribution? Here is one. And here is another.
The distribution of one of the variables.

$E[Y|X]$?

$$E[Y|X=1] = (.03\times 1 + .2\times 2 + .21\times 3 + .02\times 5)/.44 = \frac{1.16}{.44}.$$
$$E[Y|X=2] = (.05\times 1 + .01\times 2 + .06\times 3 + .2\times 5)/.32 = \frac{1.25}{.32}.$$
$$E[Y|X=4] = (.1\times 1 + .03\times 2 + .03\times 3 + .02\times 5)/.18 = \frac{.35}{.18}.$$
$$E[Y|X=8] = (.02\times 1 + .02\times 2 + .01\times 3 + .01\times 5)/.06 = \frac{.10}{.06}.$$

$$E[Y] = E[E[Y|X]] = E[Y|X=1]Pr[X=1] + E[Y|X=2]Pr[X=2] + \cdots$$

$$E[Y] = (1.16 + 1.25 + .35 + .10) = 2.86.$$

Multiple Continuous Random Variables

One defines a pair (X,Y) of continuous RVs by specifying $f_{X,Y}(x,y)$ for $x,y \in \Re$ where

$$f_{X,Y}(x,y)dxdy = Pr[X \in (x, x+dx), Y \in (y+dy)].$$

The function $f_{X,Y}(x,y)$ is called the joint pdf of X and Y.

Example: Choose a point (X,Y) uniformly in the set $A \subset \Re^2$. Then

$$f_{X,Y}(x,y) = \frac{1}{|A|} 1\{(x,y) \in A\}$$

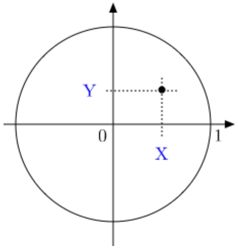
where $|A|$ is the area of A.

Interpretation. Think of (X,Y) as being discrete on a grid with mesh size ε and $Pr[X = m\varepsilon, Y = n\varepsilon] = f_{X,Y}(m\varepsilon, n\varepsilon)\varepsilon^2$.

Extension: $\mathbf{X} = (X_1, \ldots, X_n)$ with $f_{\mathbf{X}}(\mathbf{x})$.

Example of Continuous (X, Y)

Pick a point (X, Y) uniformly in the unit circle.



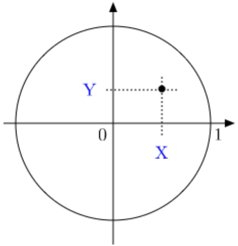
$$\implies f_{X,Y}(x,y) = \frac{1}{\pi} 1\{x^2 + y^2 \le 1\}.$$

Some events!

$$Pr[X > 0, Y > 0] = \frac{1}{4}$$
$$Pr[X < 0, Y > 0] = \frac{1}{4}$$
$$Pr[X^2 + Y^2 \le r^2] = r^2$$
$$Pr[X > Y] = \frac{1}{2}.$$

Example of Continuous (X, Y)

Pick a point (X, Y) uniformly in the unit circle.



$f_{X,Y}(x,y) = \frac{1}{\pi} 1\{x^2 + y^2 \leq 1\}$.

Marginals?

$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \frac{2}{\pi}\sqrt{1 - x^2}$

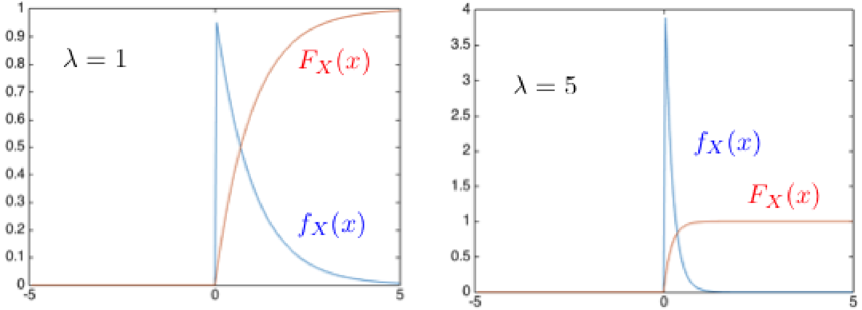
$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = \frac{2}{\pi}\sqrt{1 - y^2}$

$Expo(\lambda)$

The exponential distribution with parameter $\lambda > 0$ is defined by

$$f_X(x) = \lambda e^{-\lambda x} 1\{x \geq 0\}$$

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1 - e^{-\lambda x}, & \text{if } x \geq 0. \end{cases}$$



Note that $Pr[X > t] = e^{-\lambda t}$ for $t > 0$.

Some Properties

1. ***Expo*** **is memoryless.** Let $X = Expo(\lambda)$. Then, for $s, t > 0$,

$$\begin{aligned} Pr[X > t + s \mid X > s] &= \frac{Pr[X > t + s]}{Pr[X > s]} \\ &= \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} = e^{-\lambda t} \\ &= Pr[X > t]. \end{aligned}$$

'Used is a good as new.'

2. Scaling ***Expo*****.** Let $X = Expo(\lambda)$ and $Y = aX$ for some $a > 0$. Then

$$\begin{aligned} Pr[Y > t] &= Pr[aX > t] = Pr[X > t/a] \\ &= e^{-\lambda(t/a)} = e^{-(\lambda/a)t} = Pr[Z > t] \text{ for } Z = Expo(\lambda/a). \end{aligned}$$

Thus, $a \times Expo(\lambda) = Expo(\lambda/a)$.

Also, $Expo(\lambda) = \frac{1}{\lambda} Expo(1)$.

More Properties

3. Scaling Uniform. Let $X = U[0,1]$ and $Y = a + bX$ where $b > 0$. Then,

$$\begin{aligned}
Pr[Y \in (y, y+\delta)] &= Pr[a + bX \in (y, y+\delta)] = Pr[X \in (\frac{y-a}{b}, \frac{y+\delta-a}{b})] \\
&= Pr[X \in (\frac{y-a}{b}, \frac{y-a}{b} + \frac{\delta}{b})] = \frac{1}{b}\delta, \text{ for } 0 < \frac{y-a}{b} < 1 \\
&= \frac{1}{b}\delta, \text{ for } a < y < a+b.
\end{aligned}$$

Thus, $f_Y(y) = \frac{1}{b}$ for $a < y < a+b$. Hence, $Y = U[a, a+b]$.

Replacing b by $b-a$ we see that, if $X = U[0,1]$, then $Y = a + (b-a)X$ is $U[a,b]$.

Some More Properties

4. Scaling pdf. Let $f_X(x)$ be the pdf of X and $Y = a + bX$ where $b > 0$. Then

$$
\begin{aligned}
Pr[Y \in (y, y+\delta)] &= Pr[a + bX \in (y, y+\delta)] = Pr[X \in (\frac{y-a}{b}, \frac{y+\delta-a}{b})] \\
&= Pr[X \in (\frac{y-a}{b}, \frac{y-a}{b} + \frac{\delta}{b})] = f_X(\frac{y-a}{b})\frac{\delta}{b}.
\end{aligned}
$$

Now, the left-hand side is $f_Y(y)\delta$. Hence,

$$f_Y(y) = \frac{1}{b} f_X(\frac{y-a}{b}).$$

Expectation

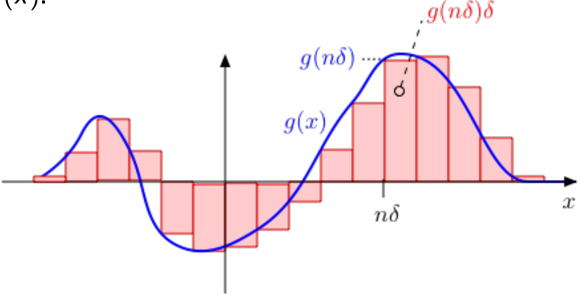
Definition: The **expectation** of a random variable X with pdf $f(x)$ is *defined* as

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx.$$

Justification: Say $X = n\delta$ w.p. $f_X(n\delta)\delta$ for $n \in \mathbb{Z}$. Then,

$$E[X] = \sum_n (n\delta) Pr[X = n\delta] = \sum_n (n\delta) f_X(n\delta)\delta = \int_{-\infty}^{\infty} x f_X(x) dx.$$

Indeed, for any g, one has $\int g(x)dx \approx \sum_n g(n\delta)\delta$. Choose $g(x) = x f_X(x)$.



Examples of Expectation

1. $X = U[0,1]$. Then, $f_X(x) = 1\{0 \le x \le 1\}$. Thus,

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx = \int_0^1 x.1 dx = [\frac{x^2}{2}]_0^1 = \frac{1}{2}.$$

2. $X =$ distance to 0 of dart shot uniformly in unit circle. Then $f_X(x) = 2x1\{0 \le x \le 1\}$. Thus,

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx = \int_0^1 x \cdot 2x dx = [\frac{2x^3}{3}]_0^1 = \frac{2}{3}.$$

Examples of Expectation

3. $X = Expo(\lambda)$. Then, $f_X(x) = \lambda e^{-\lambda x} 1\{x \geq 0\}$. Thus,

$$E[X] = \int_0^{\infty} x \lambda e^{-\lambda x} dx = -\int_0^{\infty} x de^{-\lambda x}.$$

Recall the integration by parts formula:

$$\begin{aligned} \int_a^b u(x) dv(x) &= [u(x)v(x)]_a^b - \int_a^b v(x) du(x) \\ &= u(b)v(b) - u(a)v(a) - \int_a^b v(x) du(x). \end{aligned}$$

Thus,

$$\begin{aligned} \int_0^{\infty} x de^{-\lambda x} &= [xe^{-\lambda x}]_0^{\infty} - \int_0^{\infty} e^{-\lambda x} dx \\ &= 0 - 0 + \frac{1}{\lambda} \int_0^{\infty} de^{-\lambda x} = -\frac{1}{\lambda}. \end{aligned}$$

Hence, $E[X] = \frac{1}{\lambda}$.

Independent Continuous Random Variables

Definition: The continuous RVs X and Y are independent if

$$Pr[X \in A, Y \in B] = Pr[X \in A] Pr[Y \in B], \forall A, B.$$

Theorem: The continuous RVs X and Y are independent if and only if

$$f_{X,Y}(x,y) = f_X(x) f_Y(y).$$

Proof: As in the discrete case.

Definition: The continuous RVs $X_1, \ldots, X_n$ are mutually independent if

$$Pr[X_1 \in A_1, \ldots, X_n \in A_n] = Pr[X_1 \in A_1] \cdots Pr[X_n \in A_n], \forall A_1, \ldots, A_n.$$

Theorem: The continuous RVs $X_1, \ldots, X_n$ are mutually independent if and only if

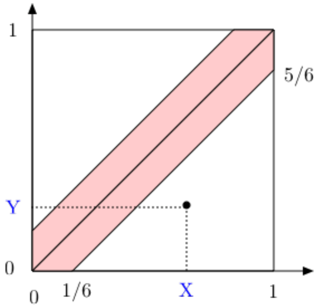
$$f_{\mathbf{X}}(x_1, \ldots, x_n) = f_{X_1}(x_1) \cdots f_{X_n}(x_n).$$

Proof: As in the discrete case.

Meeting at a Restaurant

Two friends go to a restaurant independently uniformly at random between noon and 1pm.

They agree they will wait for 10 minutes. What is the probability they meet?



Here, (X, Y) are the times when the friends reach the restaurant.

The shaded area are the pairs where $|X - Y| < 1/6$, i.e., such that they meet.

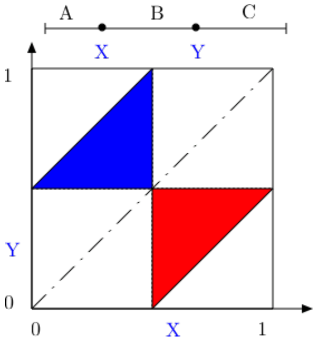
The complement is the sum of two rectangles. When you put them together, they form a square with sides $5/6$.

Thus, $Pr[\text{meet}] = 1 - (\frac{5}{6})^2 = \frac{11}{36}$.

Breaking a Stick

You break a stick at two points chosen independently uniformly at random.

What is the probability you can make a triangle with the three pieces?



Let X, Y be the two break points along the $[0,1]$ stick.

A triangle if
$A < B + C, B < A + C$, and $C < A + B$.

If $X < Y$, this means
$X < 0.5$, $Y < X + .5$, $Y > 0.5$.
This is the blue triangle.

If $X > Y$, get red triangle, by symmetry.

Thus, $Pr[\text{make triangle}] = 1/4$.

Maximum of Two Exponentials

Let $X = Expo(\lambda)$ and $Y = Expo(\mu)$ be independent.
Define $Z = \max\{X, Y\}$.

Calculate $E[Z]$.

We compute f_Z, then integrate.

One has

$$\begin{aligned} Pr[Z < z] &= Pr[X < z, Y < z] = Pr[X < z]Pr[Y < z] \\ &= (1 - e^{-\lambda z})(1 - e^{-\mu z}) = 1 - e^{-\lambda z} - e^{-\mu z} + e^{-(\lambda+\mu)z} \end{aligned}$$

Thus,

$$f_Z(z) = \lambda e^{-\lambda z} + \mu e^{-\mu z} - (\lambda + \mu)e^{-(\lambda+\mu)z}, \forall z > 0.$$

Since, $\int_0^\infty x\lambda e^{-\lambda x} dx = \lambda[-\frac{xe^{-\lambda x}}{\lambda} - \frac{e^{-\lambda x}}{\lambda^2}]_0^\infty = \frac{1}{\lambda}$.

$$E[Z] = \int_0^\infty z f_Z(z) dz = \frac{1}{\lambda} + \frac{1}{\mu} - \frac{1}{\lambda + \mu}.$$

Maximum of n i.i.d. Exponentials

Let $X_1, \ldots, X_n$ be i.i.d. *Expo*(1). Define $Z = \max\{X_1, X_2, \ldots, X_n\}$.

Calculate $E[Z]$.

We use a recursion. The key idea is as follows:

$$Z = \min\{X_1, \ldots, X_n\} + V$$

where V is the maximum of $n-1$ i.i.d. *Expo*(1). This follows from the memoryless property of the exponential.

Let then $A_n = E[Z]$. We see that

$$\begin{aligned} A_n &= E[\min\{X_1, \ldots, X_n\}] + A_{n-1} \\ &= \frac{1}{n} + A_{n-1} \end{aligned}$$

because the minimum of *Expo* is *Expo* with the sum of the rates.

Hence,

$$E[Z] = A_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} = H(n).$$

Quantization Noise

In digital video and audio, one represents a continuous value by a finite number of bits.

This introduces an error perceived as noise: the quantization noise. What is the power of that noise?

Model: $X = U[0,1]$ is the continuous value. Y is the closest multiple of 2^{-n} to X. Thus, we can represent Y with n bits. The error is $Z := X - Y$.

The power of the noise is $E[Z^2]$.

Analysis: We see that Z is uniform in $[0, a = 2^{-(n+1)}]$.

Thus,

$$E[Z^2] = \frac{a^2}{3} = \frac{1}{3}2^{-2(n+1)}.$$

The power of the signal X is $E[X^2] = \frac{1}{3}$.

Quantization Noise

We saw that $E[Z^2] = \frac{1}{3}2^{-2(n+1)}$ and $E[X^2] = \frac{1}{3}$.

The **signal to noise ratio** (SNR) is the power of the signal divided by the power of the noise.

Thus,

$$SNR = 2^{2(n+1)}.$$

Expressed in decibels, one has

$$SNR(dB) = 10\log_{10}(SNR) = 20(n+1)\log_{10}(2) \approx 6(n+1).$$

For instance, if $n = 16$, then $SNR(dB) \approx 112dB$.

Expected Squared Distance

Problem 1: Pick two points X and Y independently and uniformly at random in $[0, 1]$.

What is $E[(X - Y)^2]$?

Analysis: One has

$$\begin{aligned} E[(X-Y)^2] &= E[X^2 + Y^2 - 2XY] \\ &= \frac{1}{3} + \frac{1}{3} - 2\frac{1}{2}\frac{1}{2} \\ &= \frac{2}{3} - \frac{1}{2} = \frac{1}{6}. \end{aligned}$$

Problem 2: What about in a unit square?

Analysis: One has

$$\begin{aligned} E[||\mathbf{X} - \mathbf{Y}||^2] &= E[(X_1 - Y_1)^2] + E[(X_2 - Y_2)^2] \\ &= 2 \times \frac{1}{6}. \end{aligned}$$

Problem 3: What about in n dimensions? $\frac{n}{6}$.

Geometric and Exponential

The geometric and exponential distributions are similar. They are both memoryless.

Consider flipping a coin every $1/N$ second with $Pr[H] = p/N$, where $N \gg 1$.

Let X be the time until the first H.

Fact: $X \approx Expo(p)$.

Analysis: Note that

$$\begin{aligned} Pr[X > t] &\approx Pr[\text{first } Nt \text{ flips are tails}] \\ &= (1 - \frac{p}{N})^{Nt} \approx \exp\{-pt\}. \end{aligned}$$

Indeed, $(1 - \frac{a}{N})^N \approx \exp\{-a\}$.

Summary

Continuous Probability

- Continuous RVs are essentially the same as discrete RVs
- Think that $X \approx x$ with probability $f_X(x)\varepsilon$
- Sums become integrals, ...
- The exponential distribution is magical: memoryless.