CS70: Lecture25.

Markov Chains 1.5

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1. Review
2. Distribution
3. Irreducibility
4. Convergence

Review

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- Markov Chain:

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  - Finite set $\mathscr{X}$;

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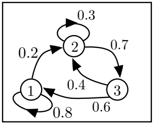
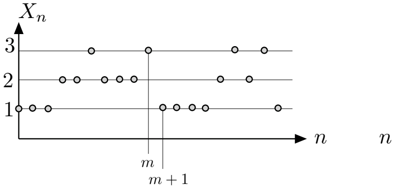
  - $\alpha(i) = \sum_j P(i,j)\alpha(j)$. $\alpha(A) = 1, \alpha(B) = 0$.

Distribution of X_n





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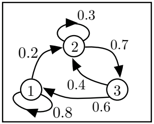

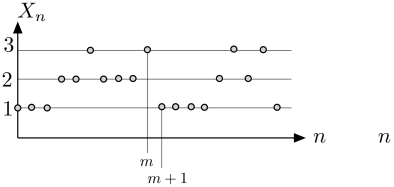

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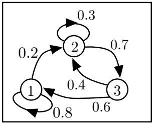

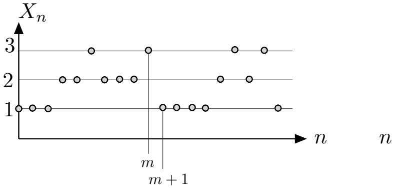

Recall π_n is a distribution over states for X_n.

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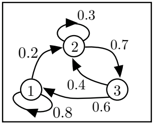

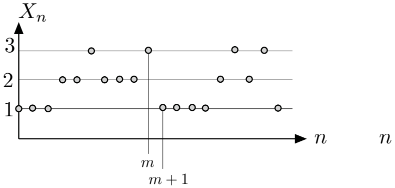

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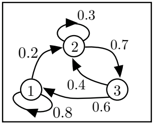

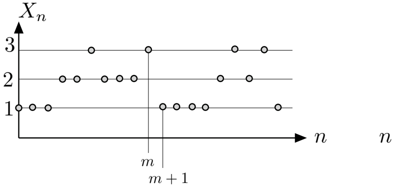

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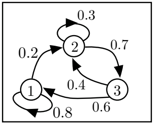

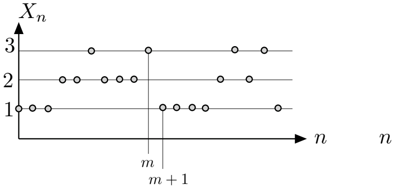

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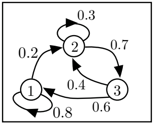

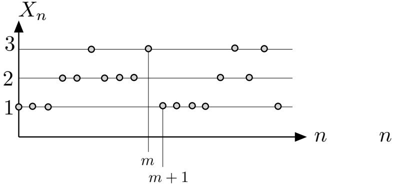

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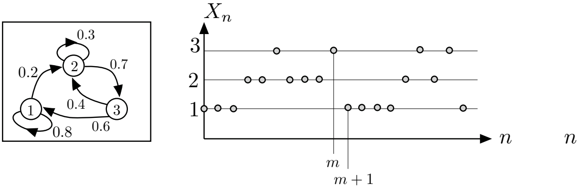

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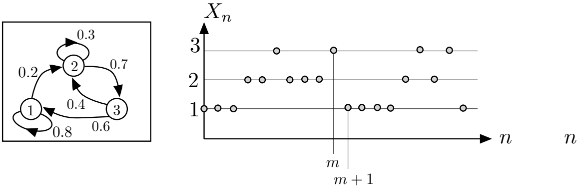

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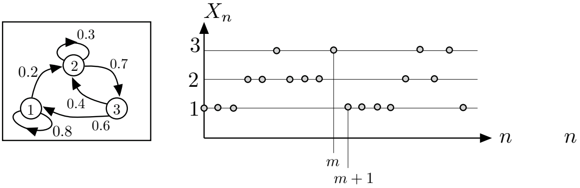

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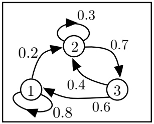

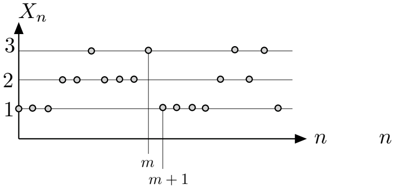

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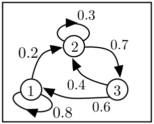

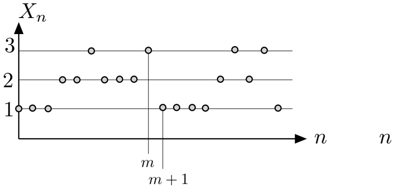

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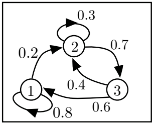

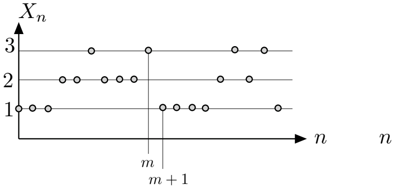

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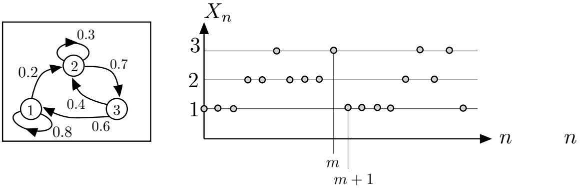

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Sometimes the distribution as $n \to \infty$

Stationary: Example

Example 1:

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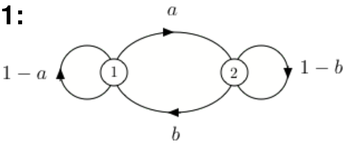

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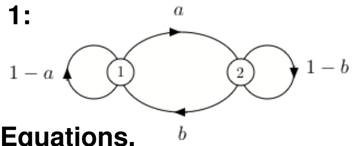

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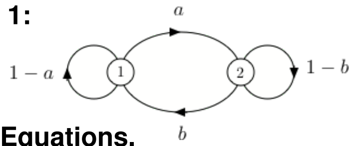

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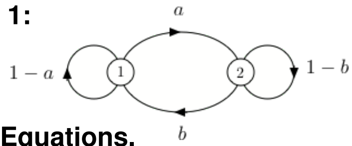

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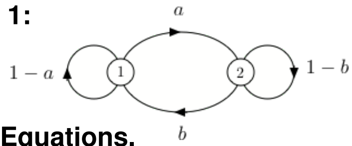

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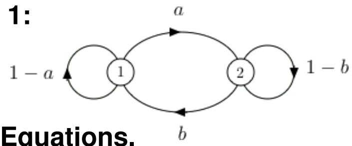

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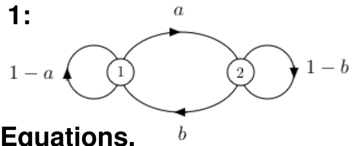

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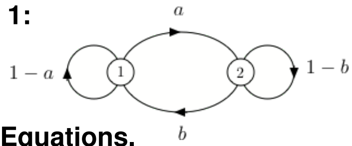

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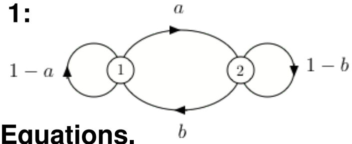

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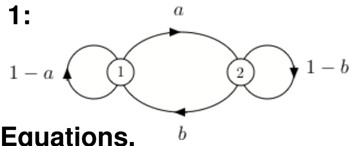

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$$

Balance Equations.

$$\begin{aligned}
\pi P = \pi \quad & \Leftrightarrow \quad [\pi(1), \pi(2)] \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} = [\pi(1), \pi(2)] \\
& \Leftrightarrow \quad \pi(1)(1-a) + \pi(2)b = \pi(1) \text{ and } \pi(1)a + \pi(2)(1-b) = \pi(2) \\
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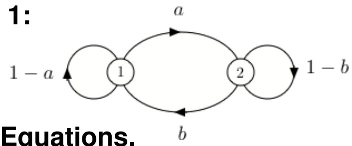

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When is here just one?

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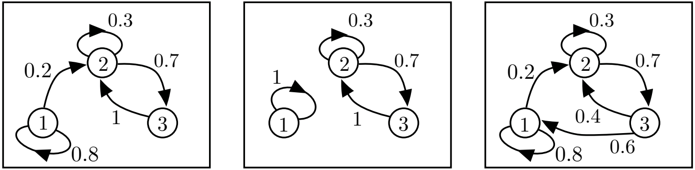

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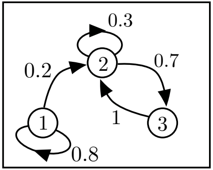

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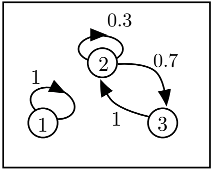

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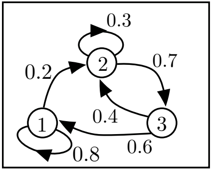

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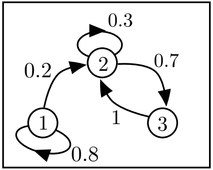

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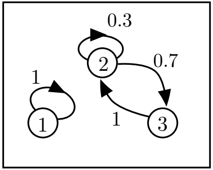

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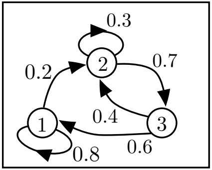

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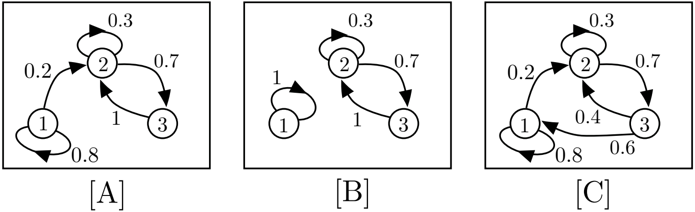

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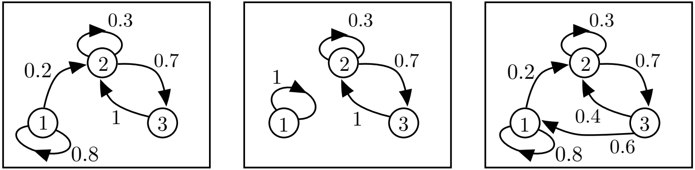

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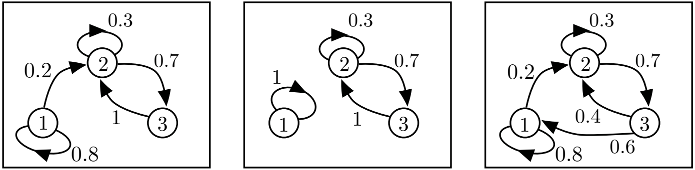

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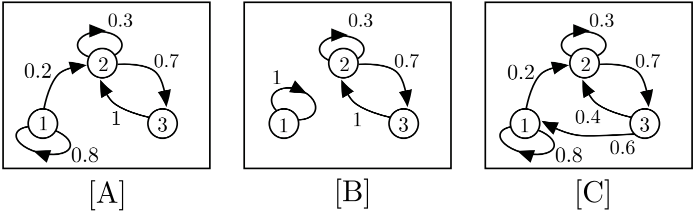

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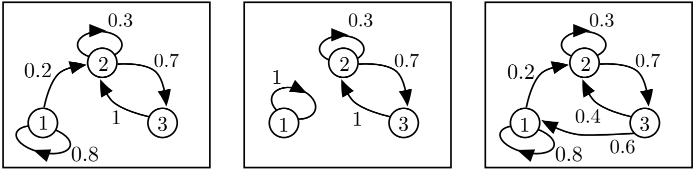

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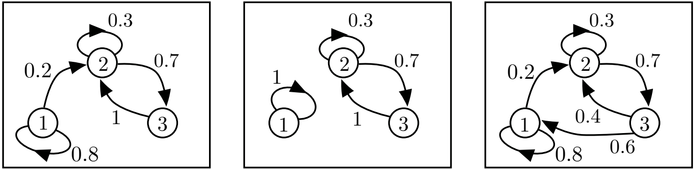

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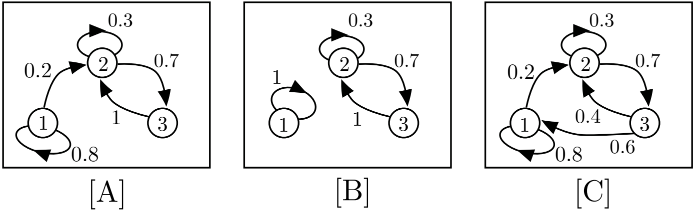

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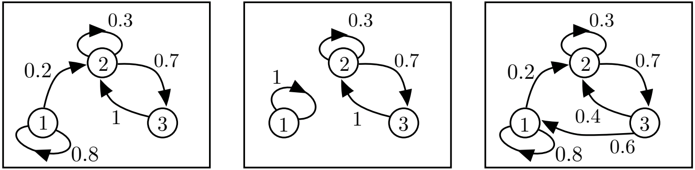

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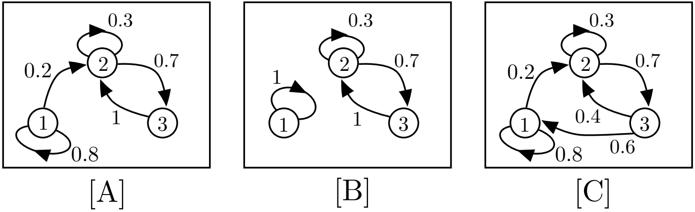

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If you consider the graph with arrows when $P(i,j) > 0$, irreducible means that there is a single connected component.

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  Only one stationary distribution if irreducible (or connected.)

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Proof: Lecture note 24 gives a plausibility argument.

□

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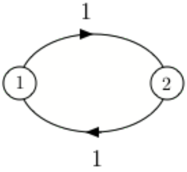

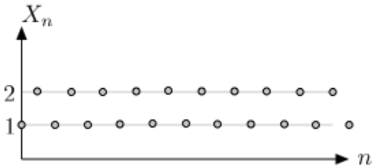

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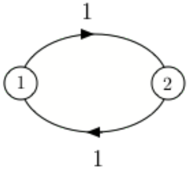

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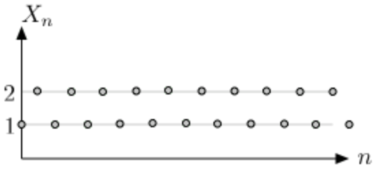

The fraction of time in state 1

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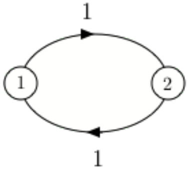

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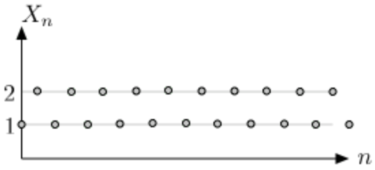

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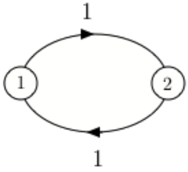

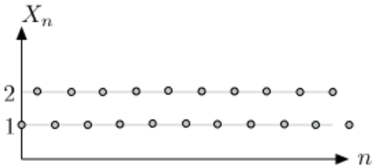

The fraction of time in state 1 converges to $1/2$, which is $\pi(1)$.

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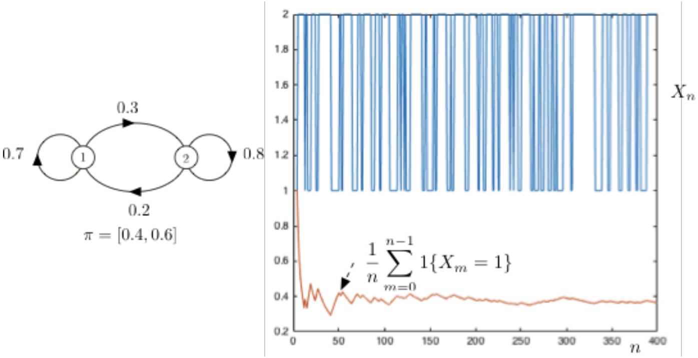

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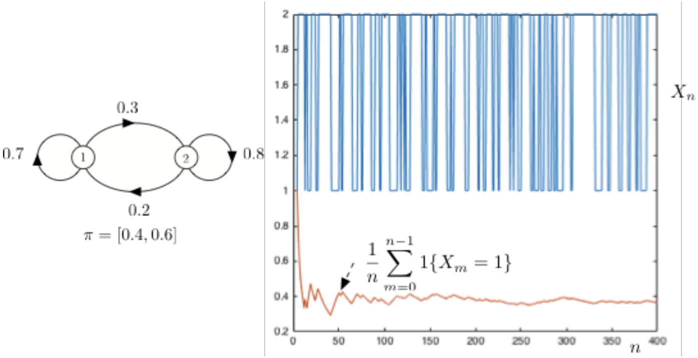

Convergence to Invariant Distribution

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Question: Assume that the MC is irreducible.

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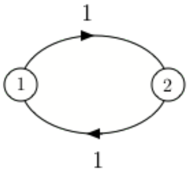

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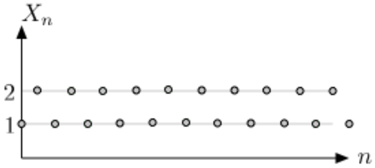

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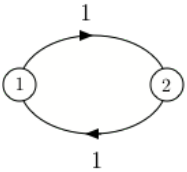

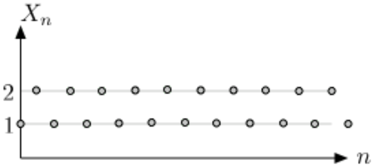

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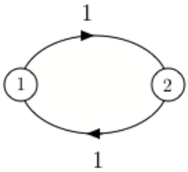

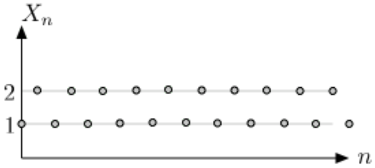

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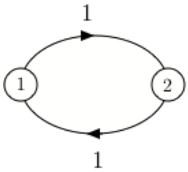

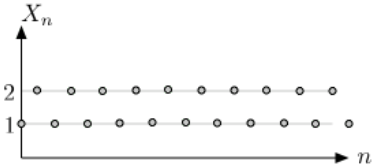

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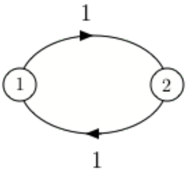

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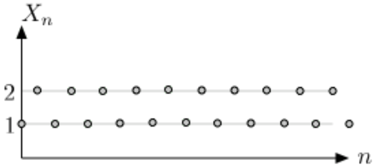

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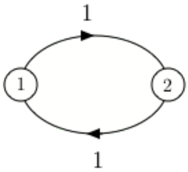

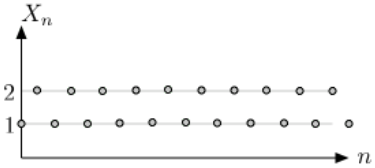

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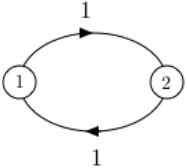

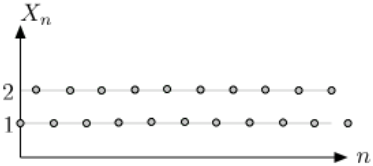

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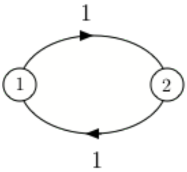

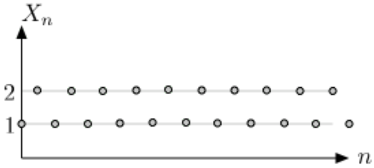

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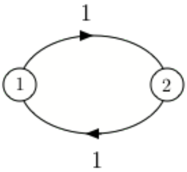

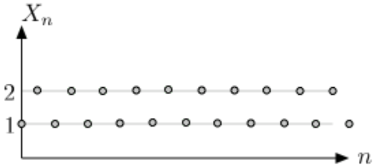

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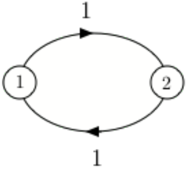

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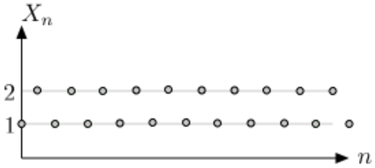

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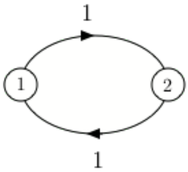

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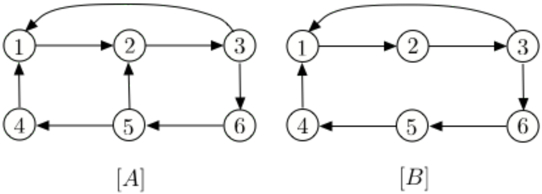

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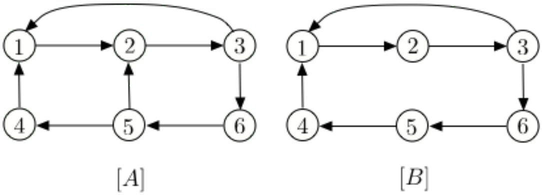

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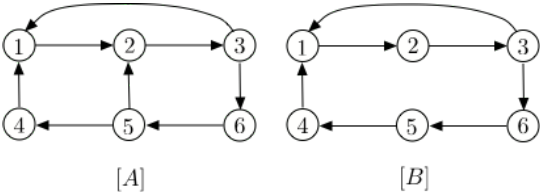

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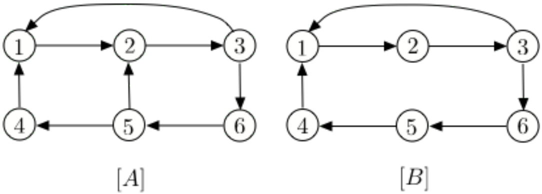

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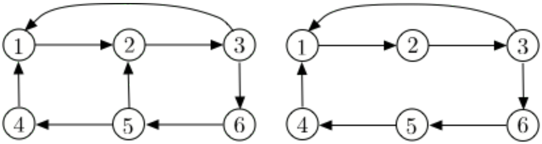

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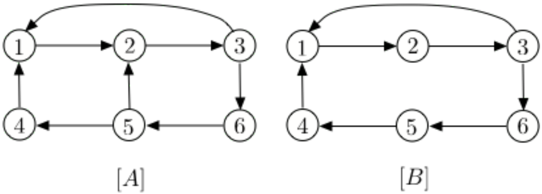

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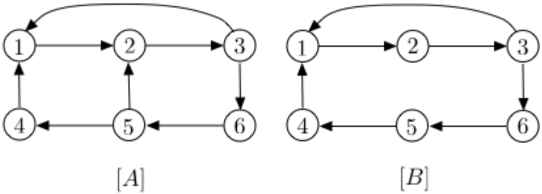

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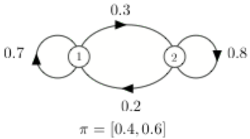

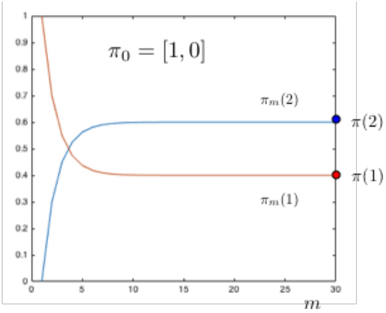

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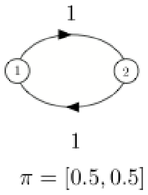

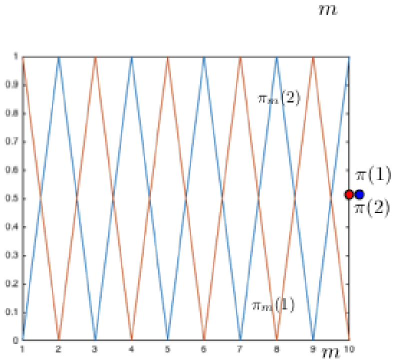

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- Calculating π: One finds $\pi = [0, 0 \ldots, 1]Q^{-1}$ where $Q = \cdots$.

CS70: Continuous Probability.

Continuous Probability 1

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1. Examples
2. Events
3. Continuous Random Variables

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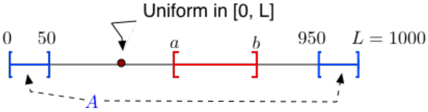

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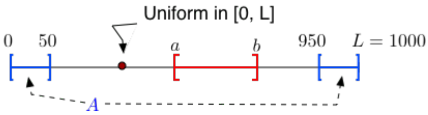

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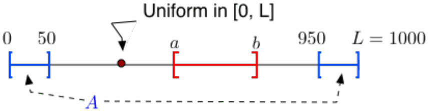

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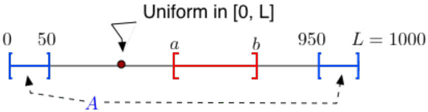

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In fact, for any $x \in [0, 1]$, one has $Pr[X = x] =$

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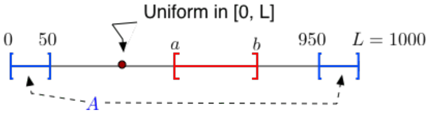

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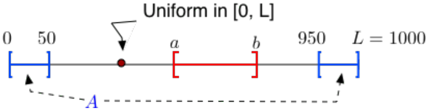

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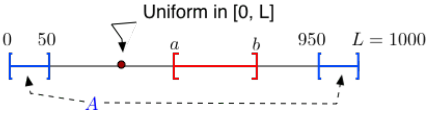

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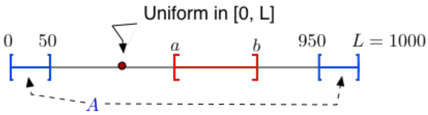

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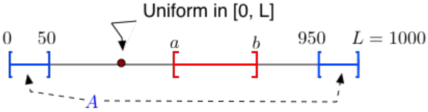

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Makes sense: $b - a$ is the fraction of $[0,1]$ that $[a,b]$ covers.

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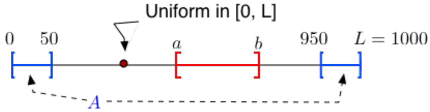

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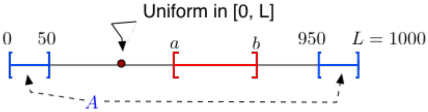

Note: A **radical** change in approach.

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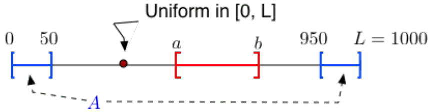

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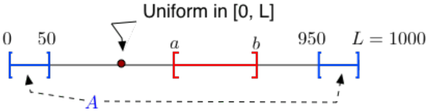

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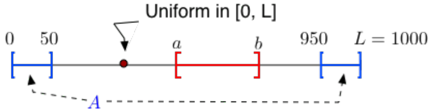

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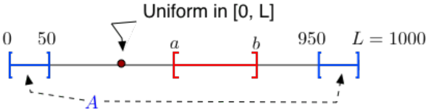

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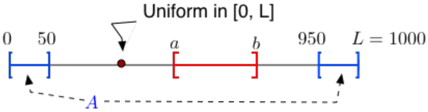

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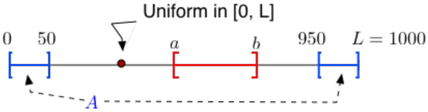

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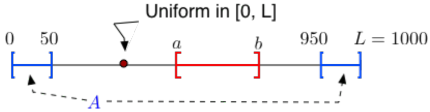

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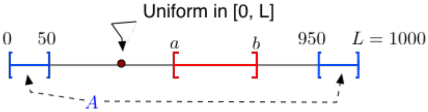

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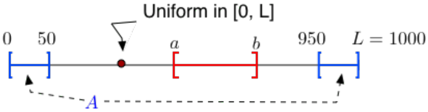

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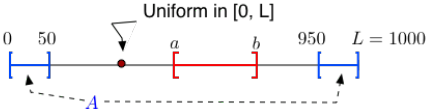

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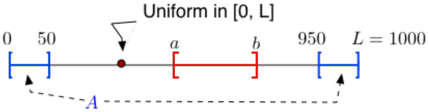

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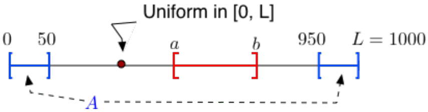

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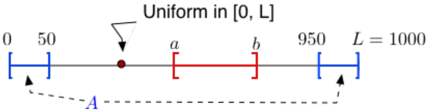

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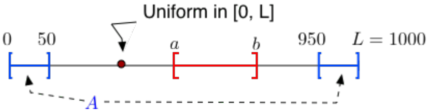

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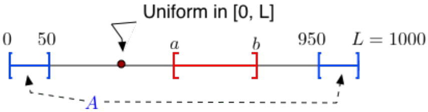

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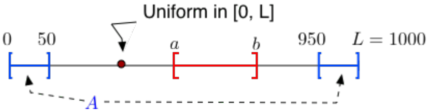

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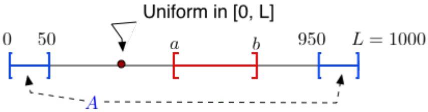

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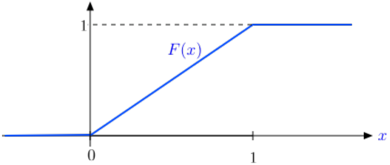

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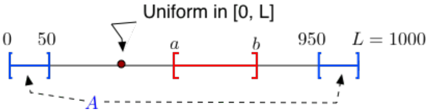

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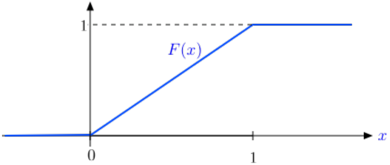

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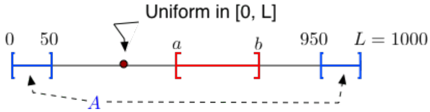

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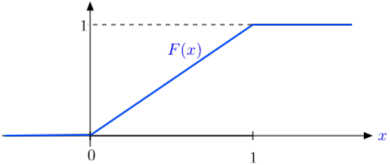

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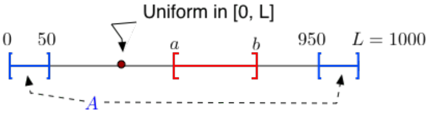

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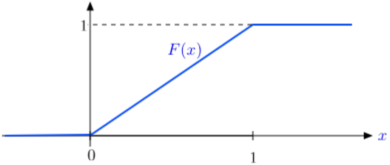

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Thus, $F(\cdot)$ specifies the probability of all the events!

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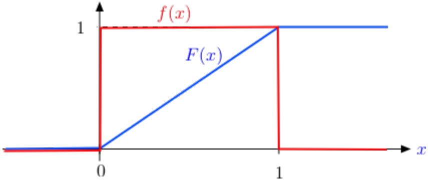

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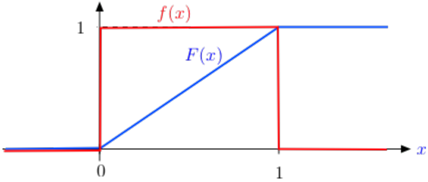

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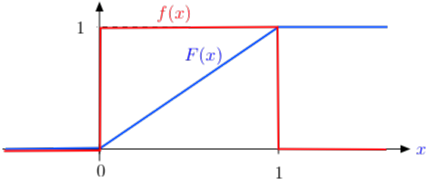

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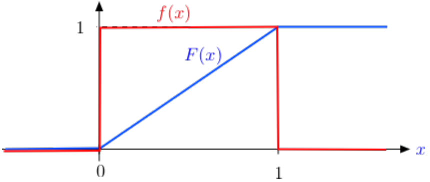
An alternative view is to define $f(x) = \frac{d}{dx} F(x) =$

Uniformly at Random in $[0, 1]$.



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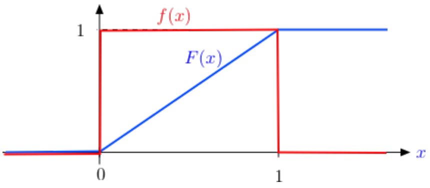

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$$F(b) - F(a) = \int_a^b f(x)dx.$$

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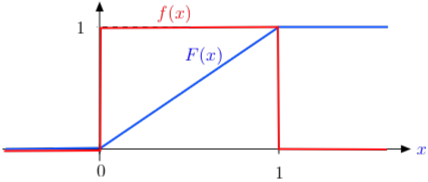

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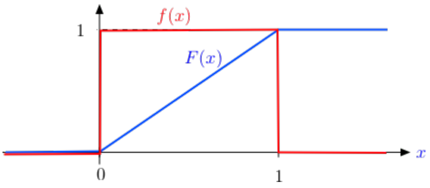

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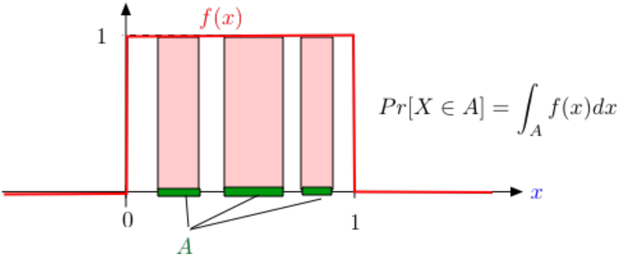
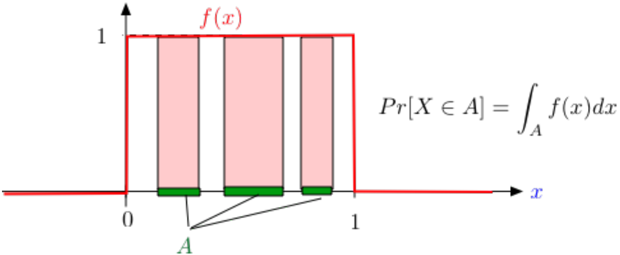
Thus, the probability of an event is the integral of $f(x)$ over the event:

$$Pr[X \in A] = \int_A f(x)dx.$$

Uniformly at Random in $[0, 1]$.

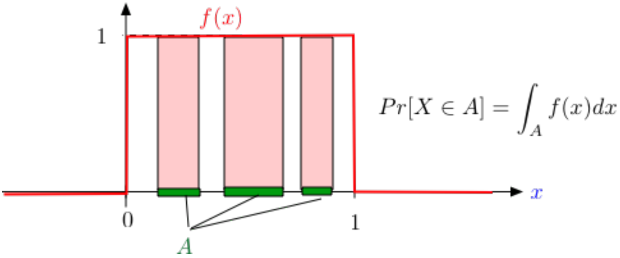


Uniformly at Random in $[0, 1]$.



Think of $f(x)$ as describing how
one unit of probability is spread over $[0, 1]$:

Uniformly at Random in $[0, 1]$.



Think of $f(x)$ as describing how
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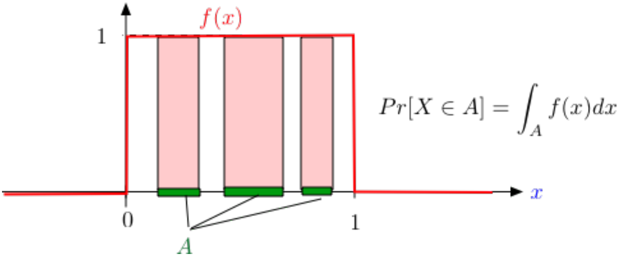
Uniformly at Random in $[0, 1]$.



Think of $f(x)$ as describing how
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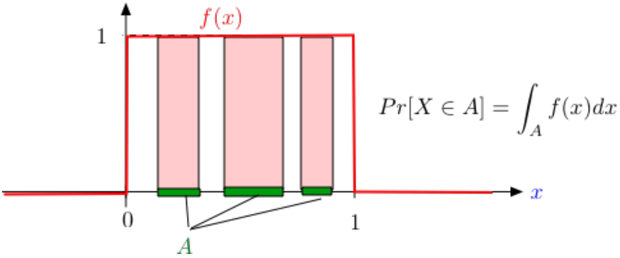
Then $Pr[X \in A]$ is the probability mass over A.

Uniformly at Random in $[0, 1]$.



Think of $f(x)$ as describing how
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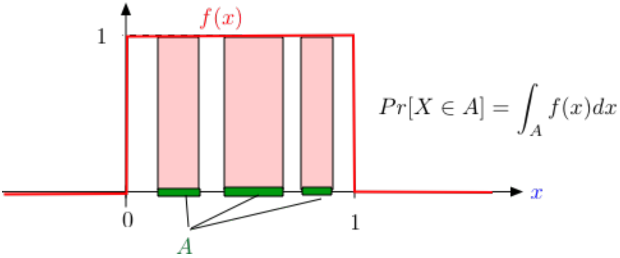
Then $Pr[X \in A]$ is the probability mass over A.

Observe:

Uniformly at Random in $[0, 1]$.



Think of $f(x)$ as describing how
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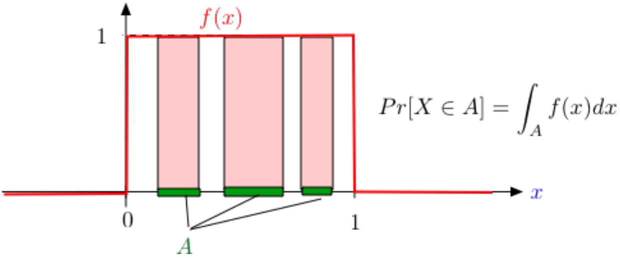
Then $Pr[X \in A]$ is the probability mass over A.

Observe:

- This makes the probability automatically additive.

Uniformly at Random in $[0, 1]$.



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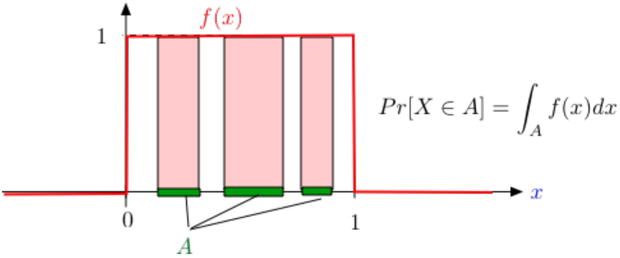

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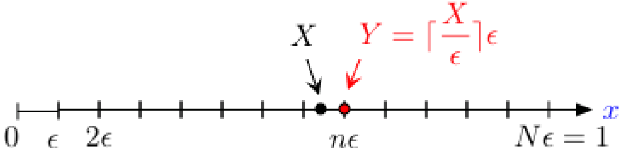
Then $Pr[X \in A]$ is the probability mass over A.

Observe:

- This makes the probability automatically additive.
- We need $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)dx = 1$.

Uniformly at Random in $[0, 1]$.

Uniformly at Random in $[0, 1]$.



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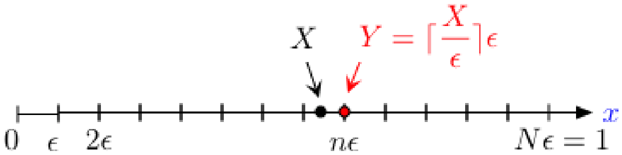
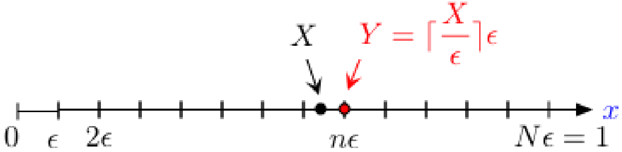


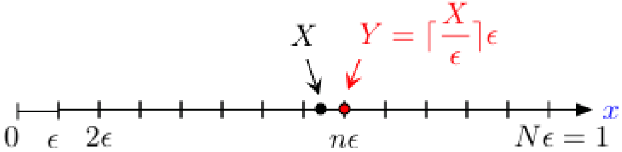
Discrete Approximation:

Uniformly at Random in $[0, 1]$.



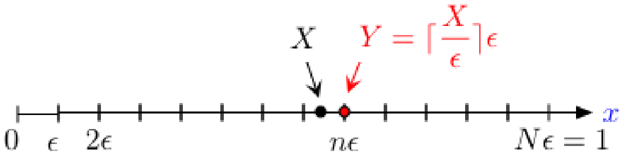
Discrete Approximation: Fix $N \gg 1$

Uniformly at Random in $[0, 1]$.



Discrete Approximation: Fix $N \gg 1$ and let $\varepsilon = 1/N$.

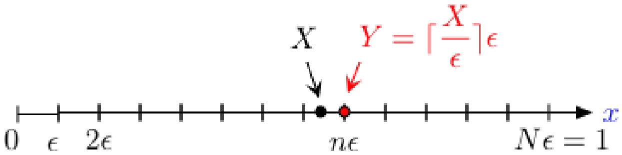
Uniformly at Random in $[0, 1]$.



Discrete Approximation: Fix $N \gg 1$ and let $\varepsilon = 1/N$.

Define $Y = n\varepsilon$ if $(n-1)\varepsilon < X \leq n\varepsilon$ for $n = 1, \ldots, N$.

Uniformly at Random in $[0,1]$.

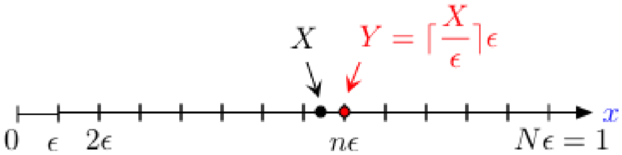


Discrete Approximation: Fix $N \gg 1$ and let $\varepsilon = 1/N$.

Define $Y = n\varepsilon$ if $(n-1)\varepsilon < X \leq n\varepsilon$ for $n = 1, \ldots, N$.

Then $|X - Y| \leq \varepsilon$

Uniformly at Random in $[0,1]$.



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Define $Y = n\varepsilon$ if $(n-1)\varepsilon < X \le n\varepsilon$ for $n = 1, \ldots, N$.

Then $|X - Y| \le \varepsilon$ and Y is discrete:

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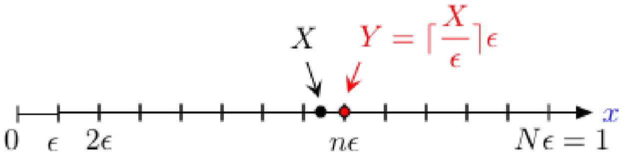

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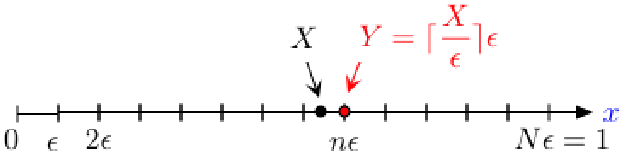
Then $|X - Y| \leq \varepsilon$ and Y is discrete: $Y \in \{\varepsilon, 2\varepsilon, \ldots, N\varepsilon\}$.

Uniformly at Random in $[0,1]$.



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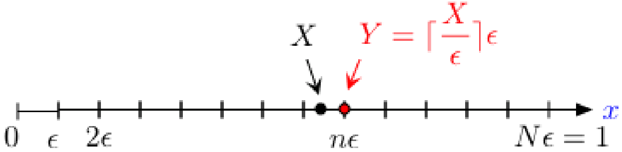
Then $|X - Y| \le \varepsilon$ and Y is discrete: $Y \in \{\varepsilon, 2\varepsilon, \ldots, N\varepsilon\}$.

Also, $Pr[Y = n\varepsilon] = \frac{1}{N}$ for $n = 1, \ldots, N$.

Uniformly at Random in $[0,1]$.



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Then $|X - Y| \leq \varepsilon$ and Y is discrete: $Y \in \{\varepsilon, 2\varepsilon, \ldots, N\varepsilon\}$.

Also, $Pr[Y = n\varepsilon] = \frac{1}{N}$ for $n = 1, \ldots, N$.

Thus, X is 'almost discrete.'

Nonuniformly at Random in $[0, 1]$.

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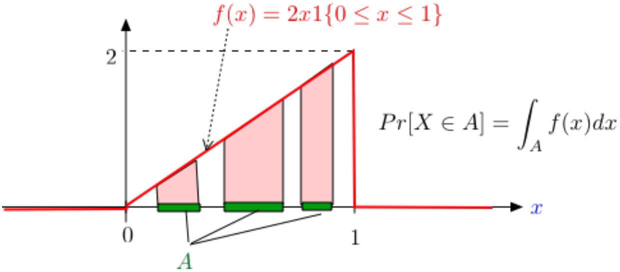

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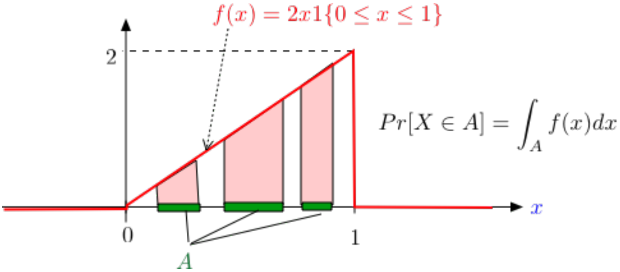
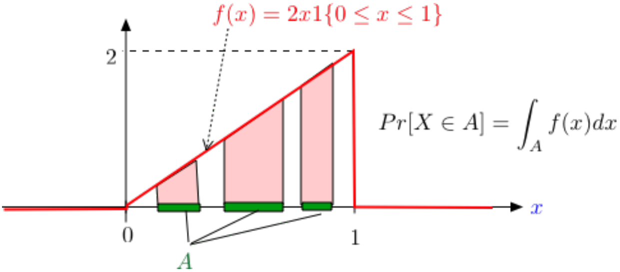


This figure shows a different choice of $f(x) \geq 0$ with $\int_{-\infty}^{\infty} f(x)dx = 1$.

Nonuniformly at Random in $[0, 1]$.



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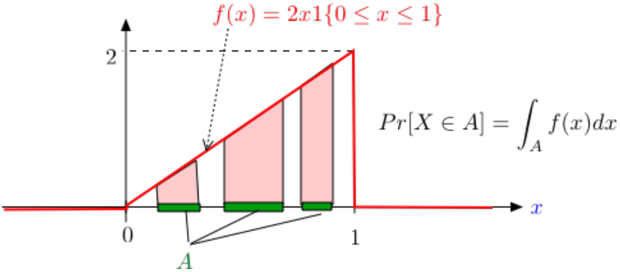

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Note that X is more likely to be closer to 1 than to 0.

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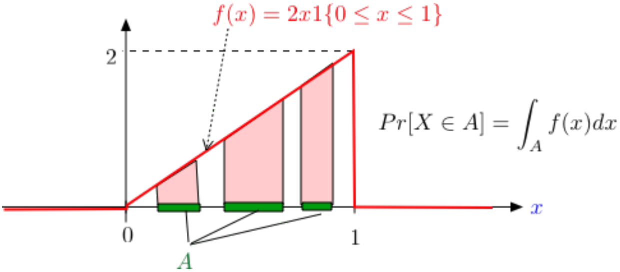

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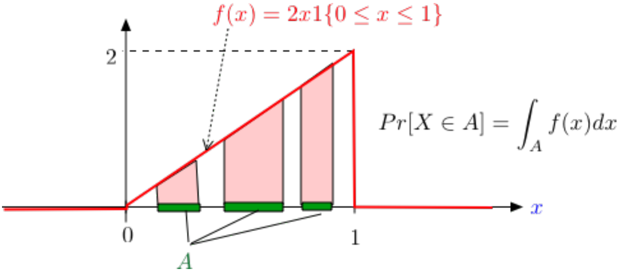

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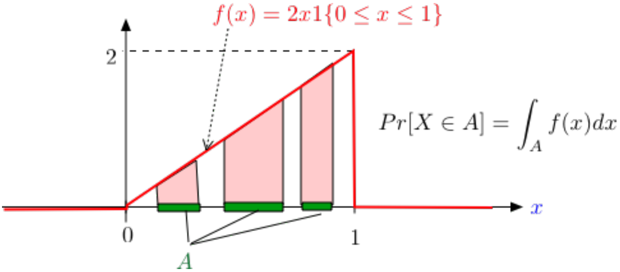
Nonuniformly at Random in $[0, 1]$.



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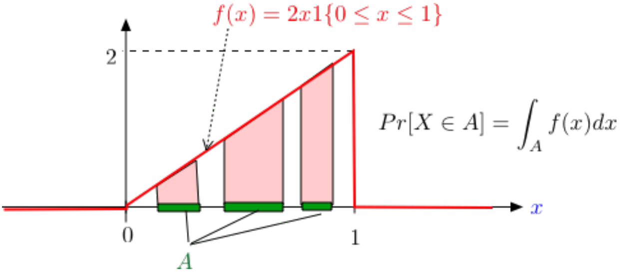

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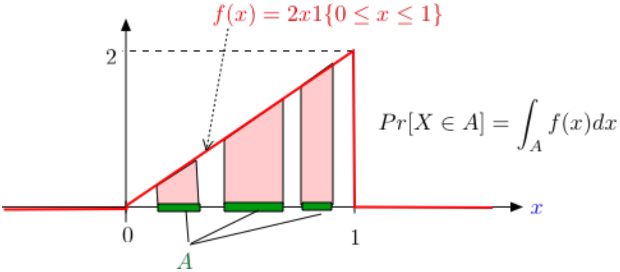
One has $Pr[X \leq x] = \int_{-\infty}^{x} f(u)du = x^2$ for $x \in [0,1]$.

Also, $Pr[X \in (x, x+\varepsilon)] = \int_{x}^{x+\varepsilon} f(u)du$

Nonuniformly at Random in $[0, 1]$.



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Another Nonuniform Choice at Random in $[0, 1]$.

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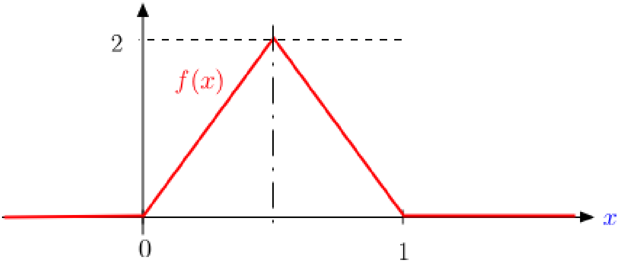

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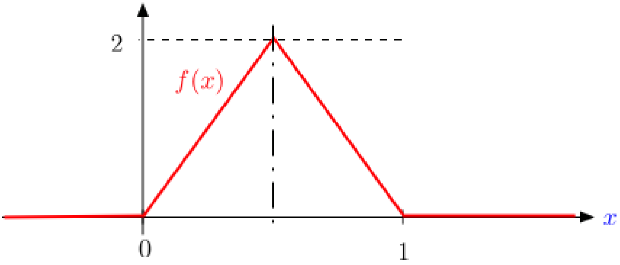

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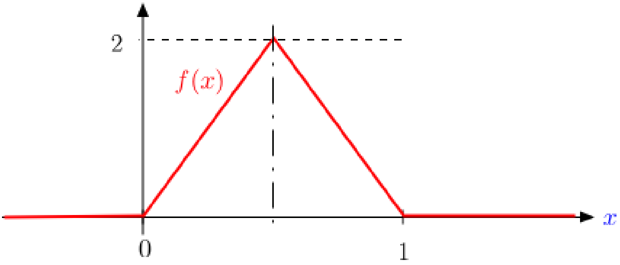

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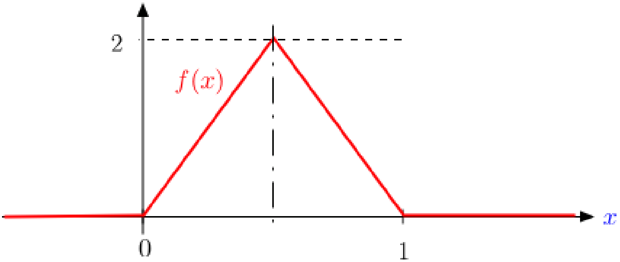

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Note that X is more likely to be closer to $1/2$ than to 0 or 1.

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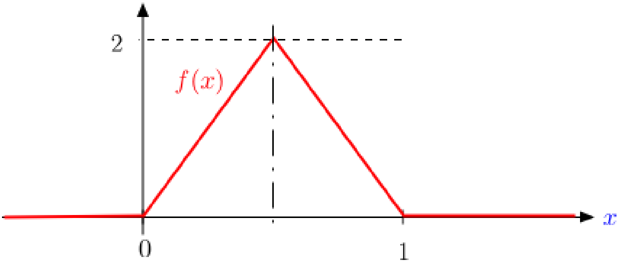

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For instance, $Pr[X \in [0, 1/3]] =$

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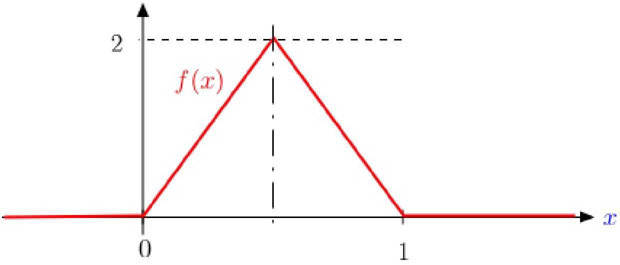

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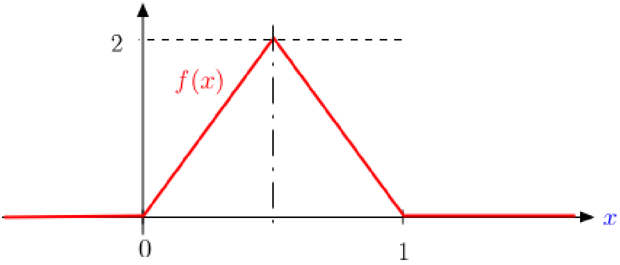

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For instance, $Pr[X \in [0, 1/3]] = \int_0^{1/3} 4xdx = 2\left[x^2\right]_0^{1/3} = \frac{2}{9}$.

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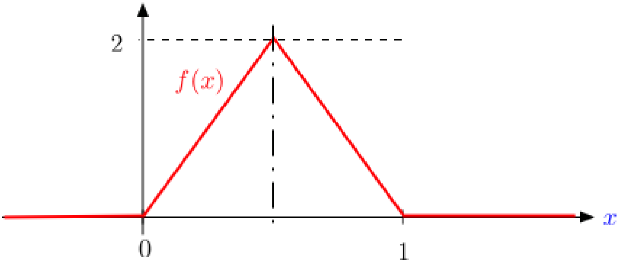

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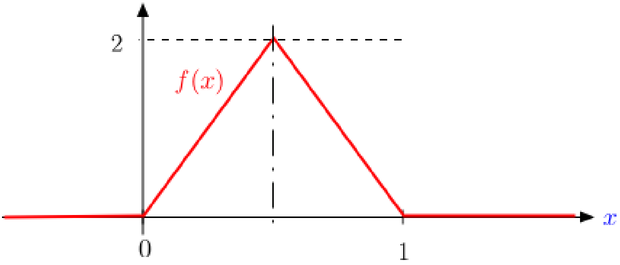
Thus, $Pr[X \in [0, 1/3]] = Pr[X \in [2/3, 1]] = \frac{2}{9}$

Another Nonuniform Choice at Random in $[0, 1]$.



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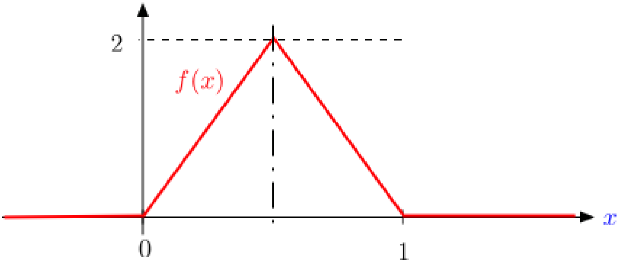
Thus, $Pr[X \in [0, 1/3]] = Pr[X \in [2/3, 1]] = \frac{2}{9}$ and $Pr[X \in [1/3, 2/3]] =$

Another Nonuniform Choice at Random in $[0, 1]$.



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It defines another way of choosing X at random in $[0, 1]$.

Note that X is more likely to be closer to $1/2$ than to 0 or 1.

For instance, $Pr[X \in [0, 1/3]] = \int_0^{1/3} 4xdx = 2\left[x^2\right]_0^{1/3} = \frac{2}{9}$.

Thus, $Pr[X \in [0, 1/3]] = Pr[X \in [2/3, 1]] = \frac{2}{9}$ and $Pr[X \in [1/3, 2/3]] = \frac{5}{9}$.

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To indicate that F and f correspond to the RV X, we will write them $F_X(x)$ and $f_X(x)$.

$$Pr[X \in (x, x+\varepsilon)]$$

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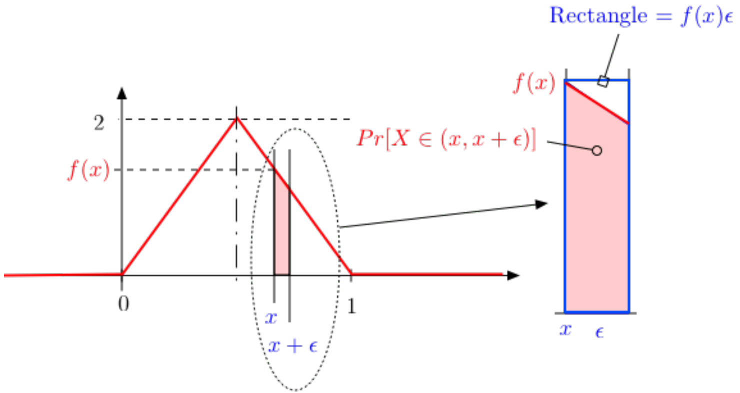

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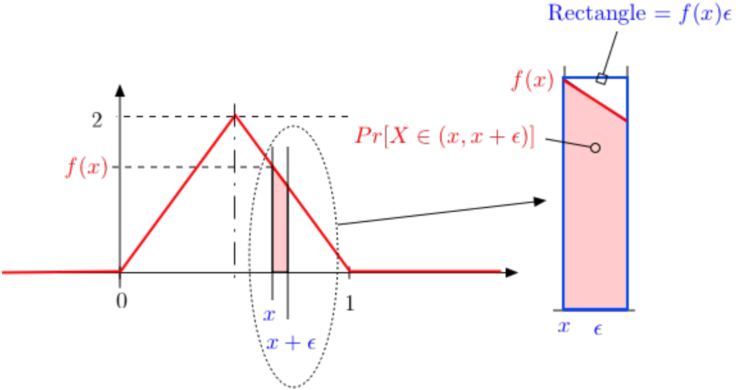

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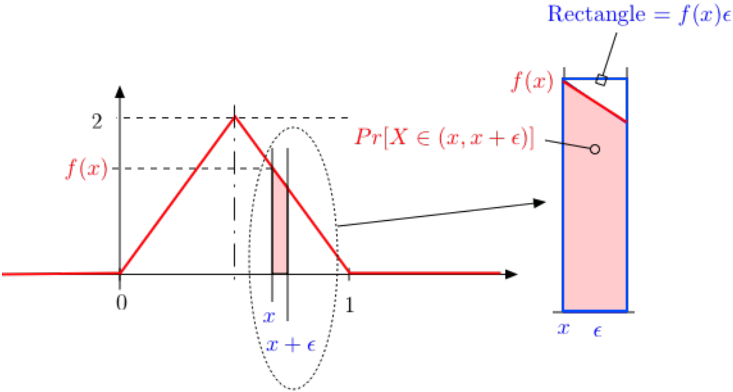

Thus, the pdf is the ‘local probability by unit length.’

It is the ‘probability density.’

Discrete Approximation

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Example: CDF

Example: hitting random location on gas tank.

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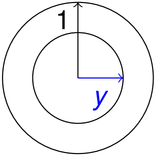

Random Variable: Y distance from center.

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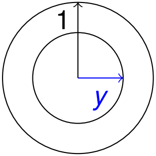

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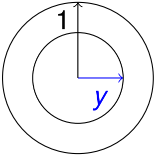

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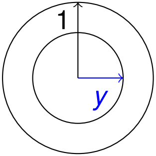

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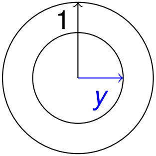

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Hence,

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Calculation of event with dartboard..

Probability between .5 and .6 of center?

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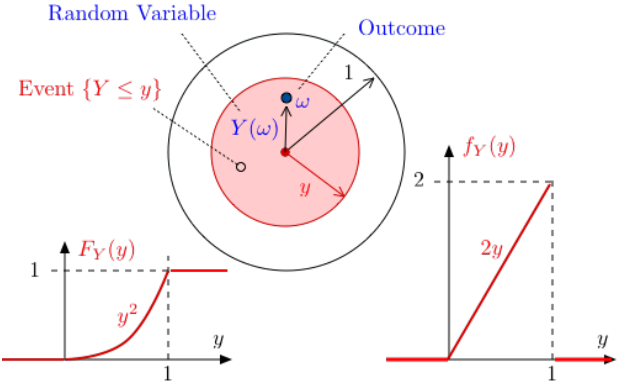
The cumulative distribution function (cdf) and probability distribution function (pdf) give full information.
Use whichever is convenient.

Target

Target



Random Variable
Outcome
Event $\{Y \leq y\}$
1
ω
$Y(\omega)$
y
$f_Y(y)$
2
$2y$
y
1
$F_Y(y)$
1
y^2
y
1

$U[a, b]$

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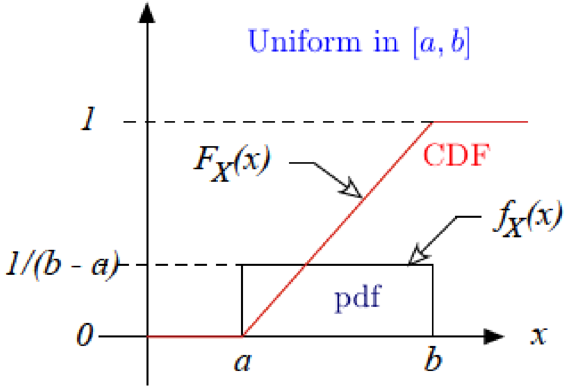

Expo(λ)

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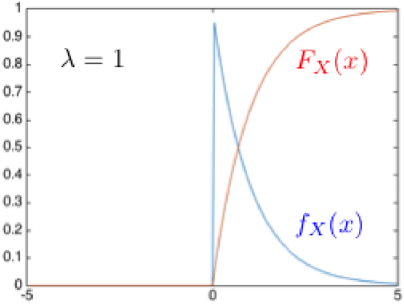

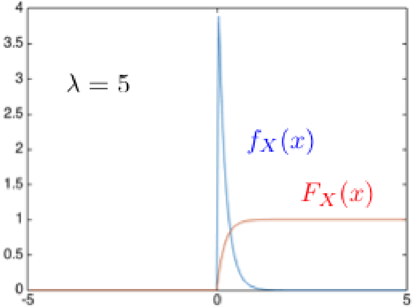

$Expo(\lambda)$

The exponential distribution with parameter $\lambda > 0$ is defined by

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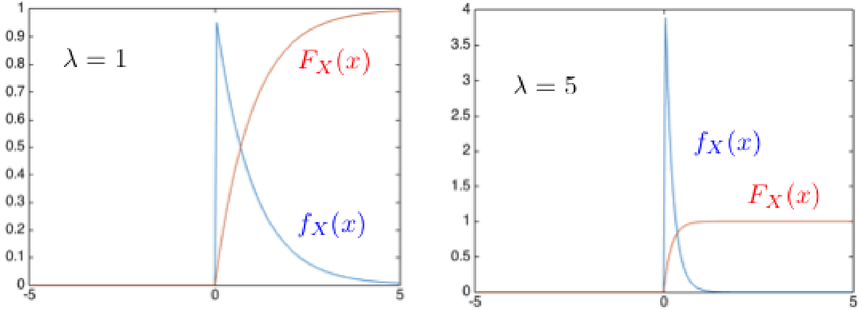

Note that $Pr[X > t] = e^{-\lambda t}$ for $t > 0$.

Continuous Random Variables

Continuous random variable X, specified by

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   1.1 $0 \leq F_X(x) \leq 1$ for all $x \in \Re$.

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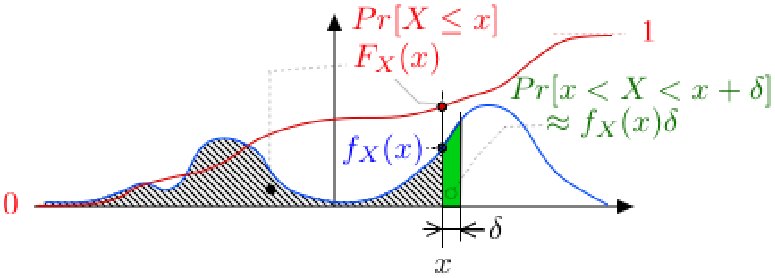
X "takes" value $n\delta$, for $n \in Z$, with $Pr[X = n\delta] = f_X(n\delta)\delta$

A Picture



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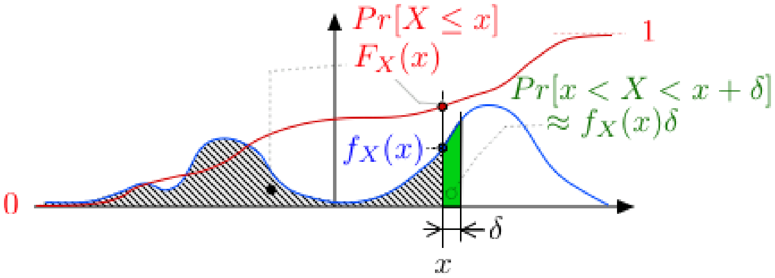

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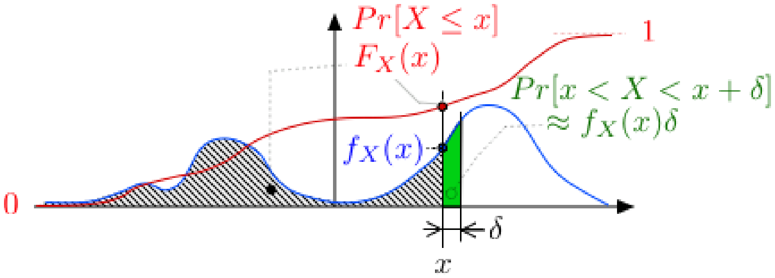

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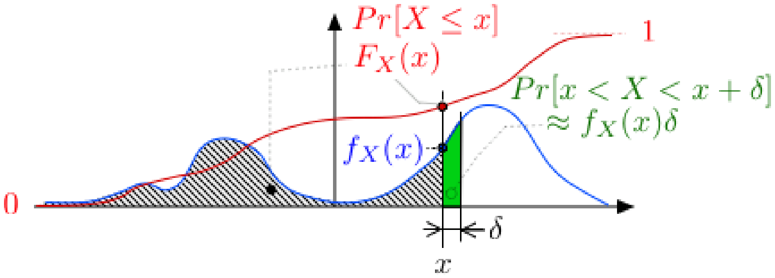

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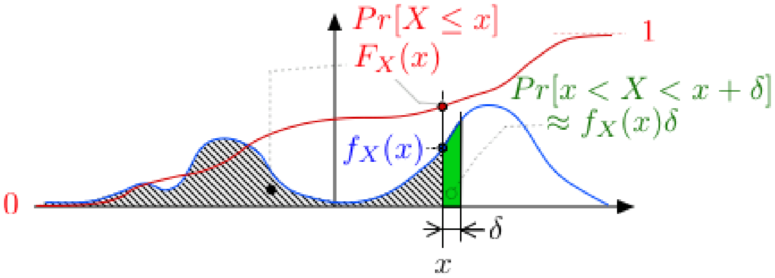

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Extension: $\mathbf{X} = (X_1, \ldots, X_n)$ with $f_{\mathbf{X}}(\mathbf{x})$.

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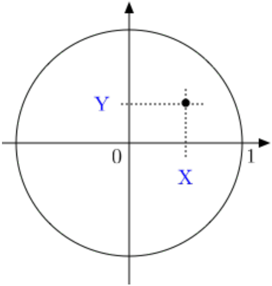

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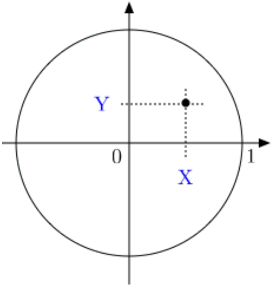

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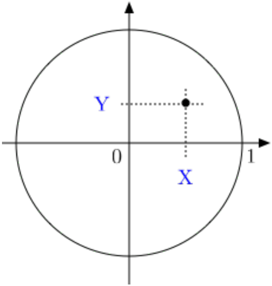

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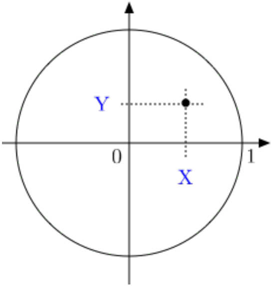

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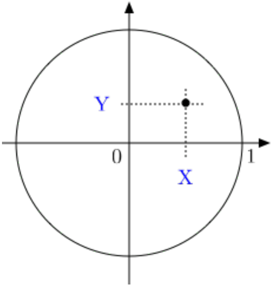

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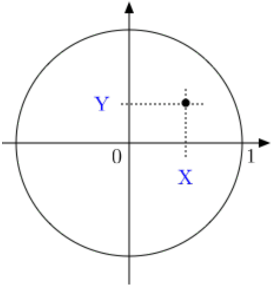

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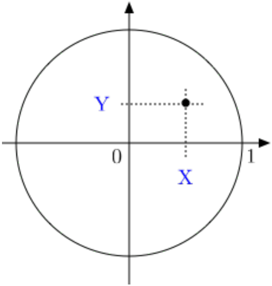

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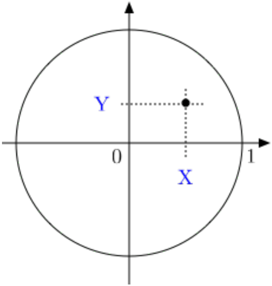

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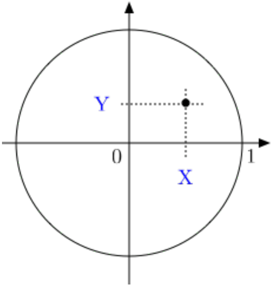

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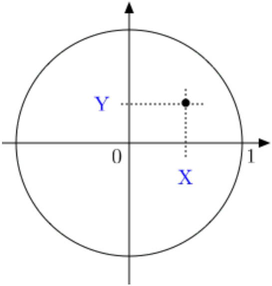

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3. $U[a,b]$: $f_X(x) = \frac{1}{b-a} 1\{a \leq x \leq b\}; F_X(x) = \frac{x-a}{b-a}$ for $a \leq x \leq b$.
4. $Expo(\lambda)$:
   $f_X(x) = \lambda \exp\{-\lambda x\} 1\{x \geq 0\}; F_X(x) = 1 - \exp\{-\lambda x\}$ for $x \leq 0$.

Summary

Continuous Probability 1

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5. Target:

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5. Target: $f_X(x) = 2x1\{0 \leq x \leq 1\}$;

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5. Target: $f_X(x) = 2x1\{0 \leq x \leq 1\}; F_X(x) = x^2$ for $0 \leq x \leq 1$.
6. Joints:

Summary

Continuous Probability 1

1. pdf: $Pr[X \in (x, x+\delta]] = f_X(x)\delta$.
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6. Joints: Is this 4/20?

Summary

Continuous Probability 1

1. pdf: $Pr[X \in (x, x+\delta]] = f_X(x)\delta$.
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4. $Expo(\lambda)$:
   $f_X(x) = \lambda \exp\{-\lambda x\}1\{x \geq 0\}; F_X(x) = 1 - \exp\{-\lambda x\}$ for $x \leq 0$.
5. Target: $f_X(x) = 2x1\{0 \leq x \leq 1\}; F_X(x) = x^2$ for $0 \leq x \leq 1$.
6. Joints: Is this 4/20?
   Joint pdf: $Pr[X \in (x, x+\delta), Y = (y, y+\delta)) = f_{X,Y}(x,y)\delta^2$.