Today

Finish up Conditional Expectation.

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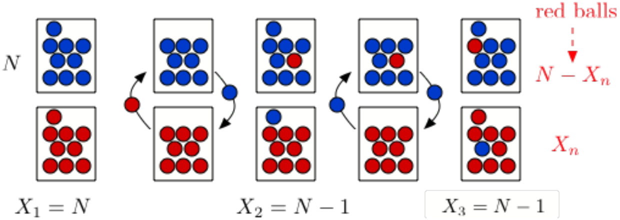
Markov Chains.

Application: Mixing



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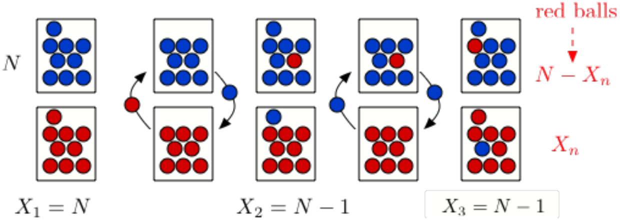

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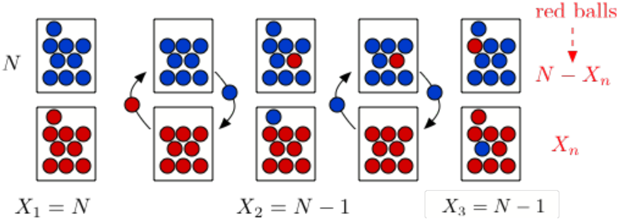

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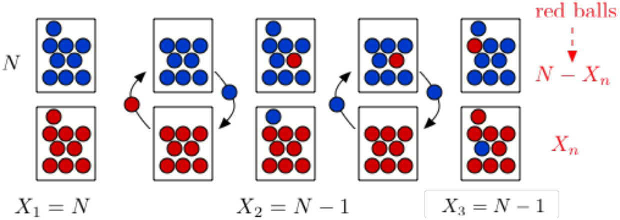

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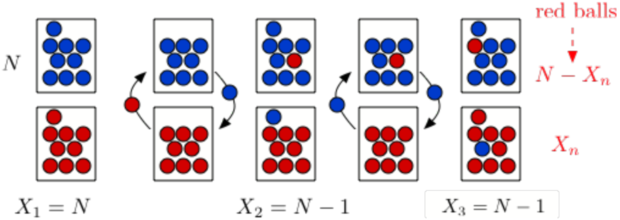

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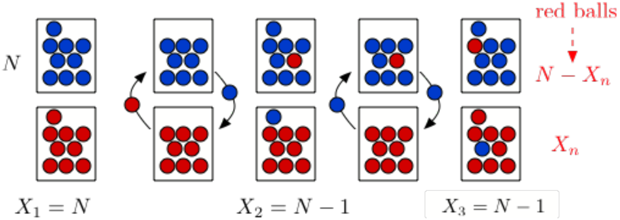

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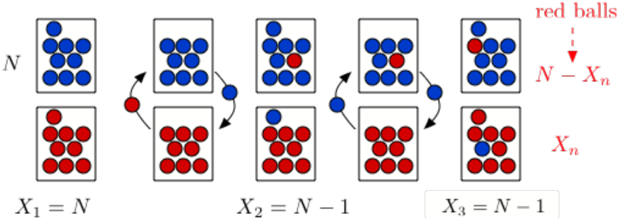

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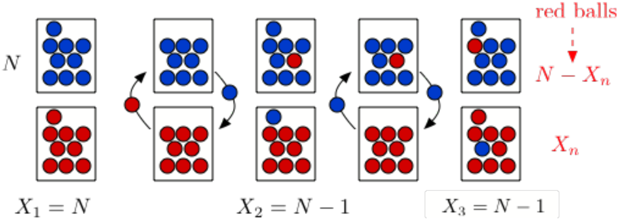

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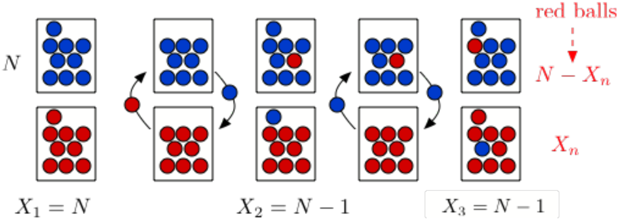

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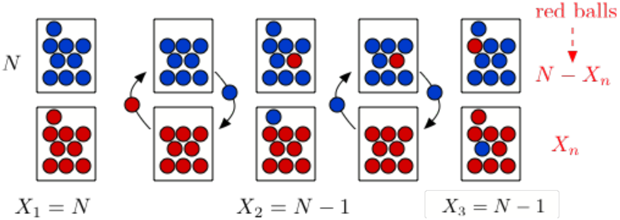

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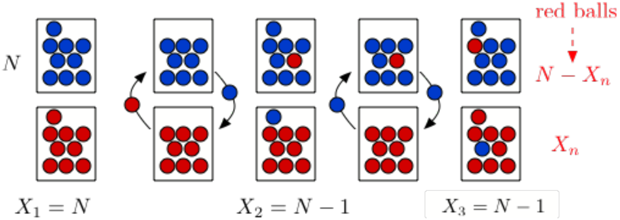

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Since $1 - \rho = 2/N$. And $\rho^n \to 0$.

Application: Mixing

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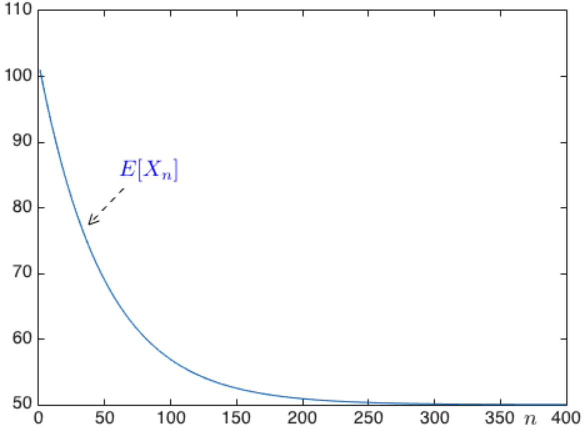

Application: Going Viral

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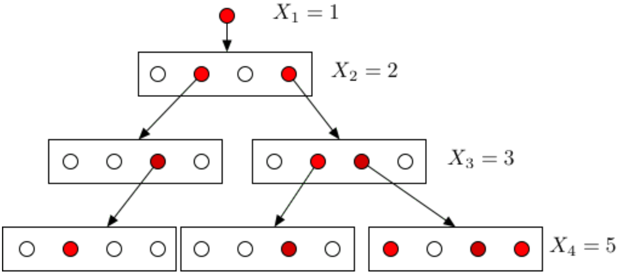

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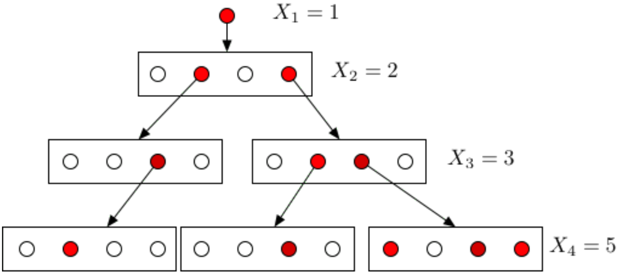

In this example, $d = 4$.

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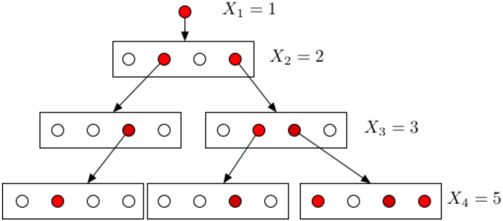

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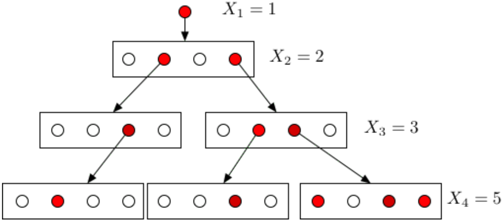
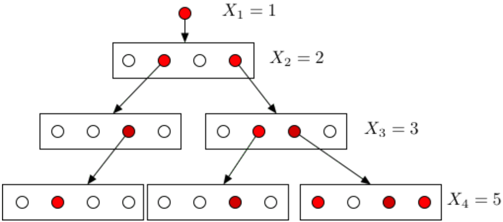


Fact:

Application: Going Viral



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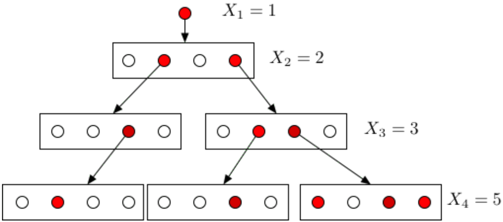

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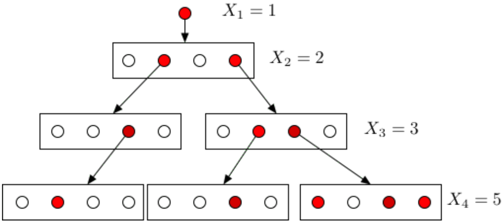

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Given $X_n = k$, $X_{n+1} = B(kd, p)$.

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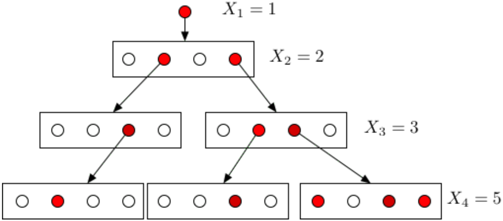

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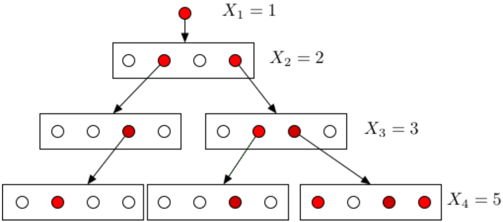

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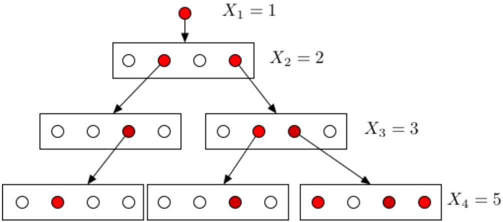

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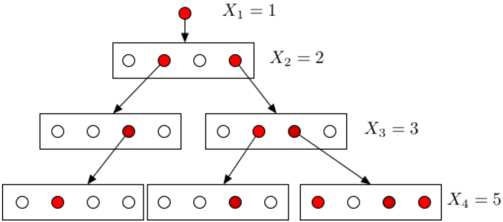

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If $pd < 1$, then $E[X_1 + \cdots + X_n] \leq (1 - pd)^{-1}$

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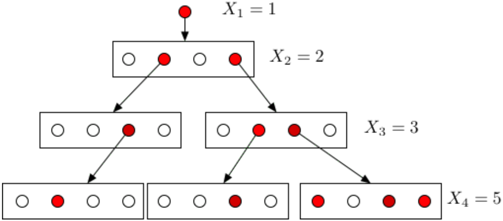

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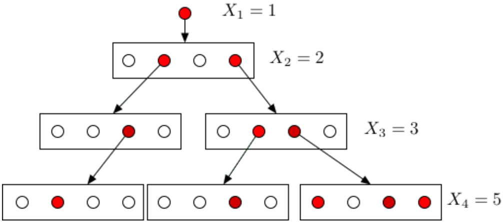

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If $pd \geq 1$, then for all C one can find n s.t.

$$E[X] \geq E[X_1 + \cdots + X_n] \geq C.$$

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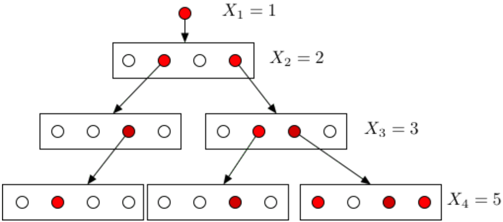

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□

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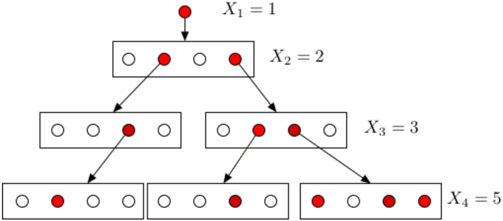


Fact: Number of tweets $X = \sum_{n=1}^{\infty} X_n$ where X_n is tweets in level n. Then, $E[X] < \infty$ iff $pd < 1$.

Proof:
Given $X_n = k$, $X_{n+1} = B(kd, p)$. Hence, $E[X_{n+1}|X_n = k] = kpd$.

Thus, $E[X_{n+1}|X_n] = pdX_n$. Consequently, $E[X_n] = (pd)^{n-1}, n \geq 1$.

If $pd < 1$, then $E[X_1 + \cdots + X_n] \leq (1 - pd)^{-1} \implies E[X] \leq (1 - pd)^{-1}$.

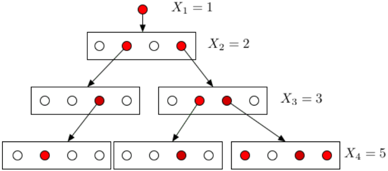
If $pd \geq 1$, then for all C one can find n s.t.

$$E[X] \geq E[X_1 + \cdots + X_n] \geq C.$$

□

In fact, one can show that $pd \geq 1 \implies Pr[X = \infty] > 0$.

Application: Going Viral



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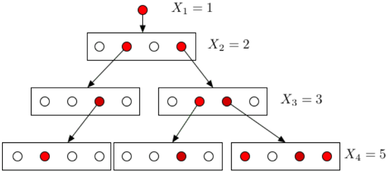

An easy extension:

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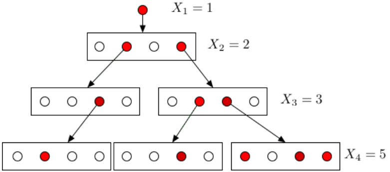

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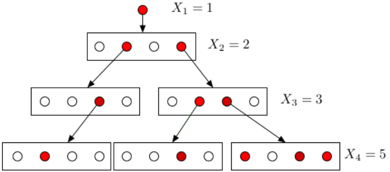

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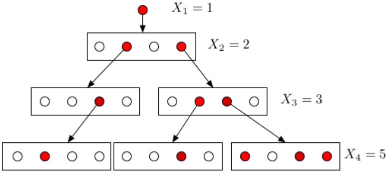

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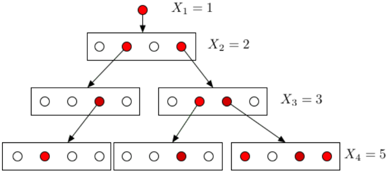
Why?

Application: Going Viral



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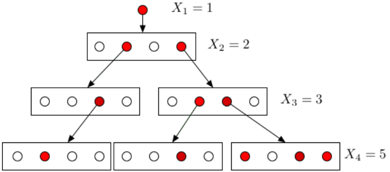
Application: Going Viral



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Why? Given $X_n = k$.
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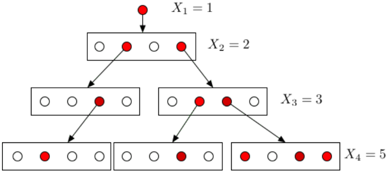

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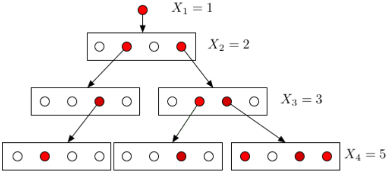

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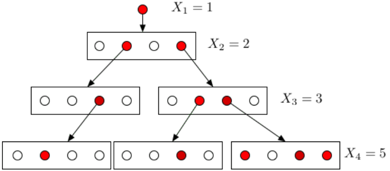

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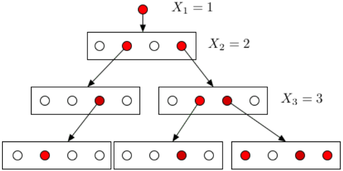

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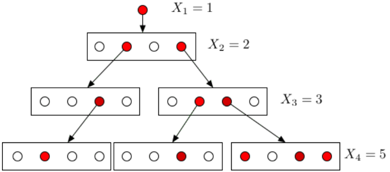

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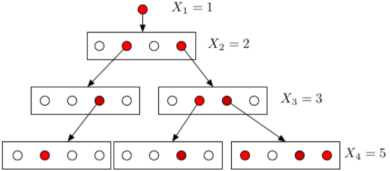

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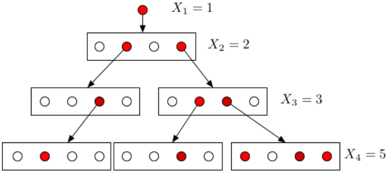

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We conclude as before.

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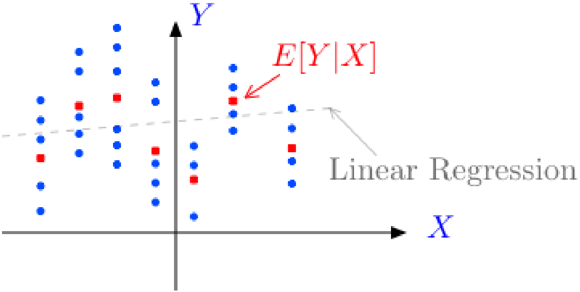

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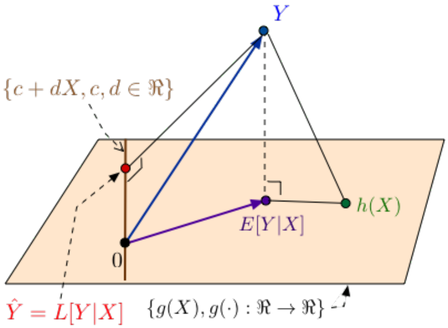
$E[Y|X]$ and $L[Y|X]$ as projections



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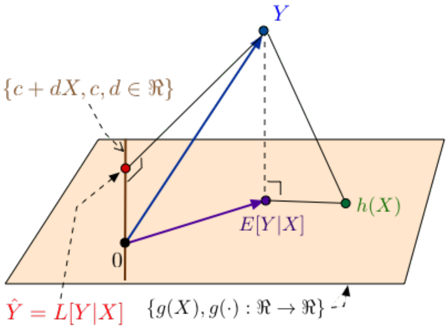

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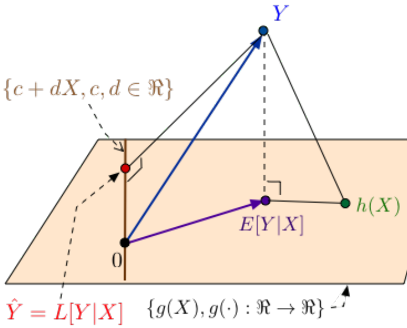

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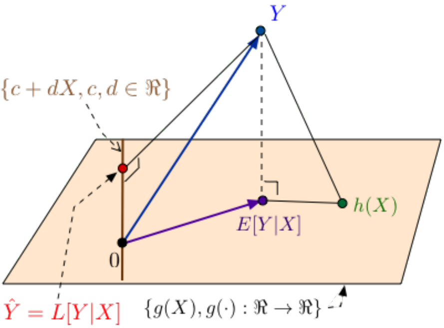

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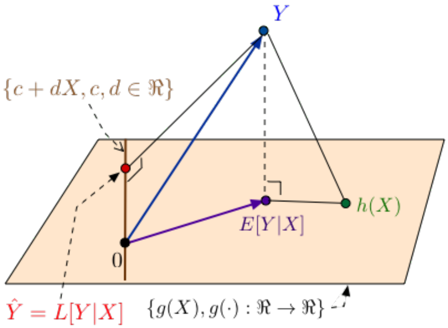

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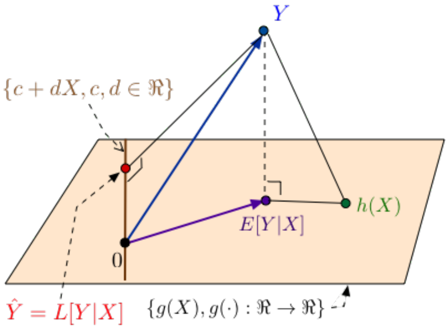

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Functions of X are linear subspace?

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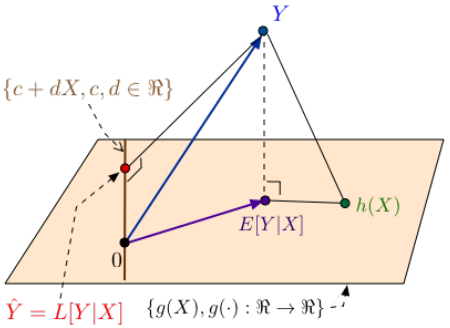

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Vector $(g(X(\omega_1), \ldots, g(X(\omega_\Omega)).$

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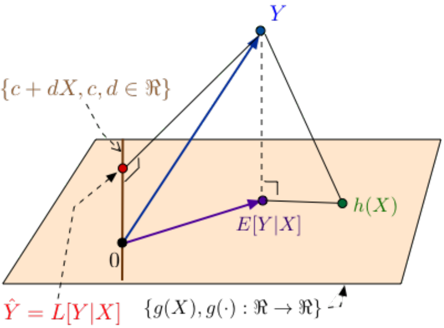

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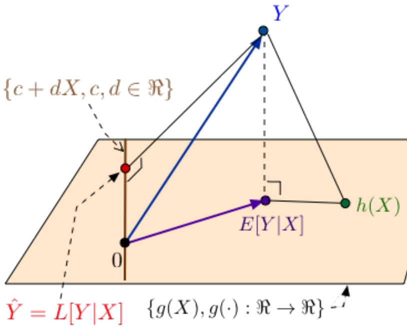

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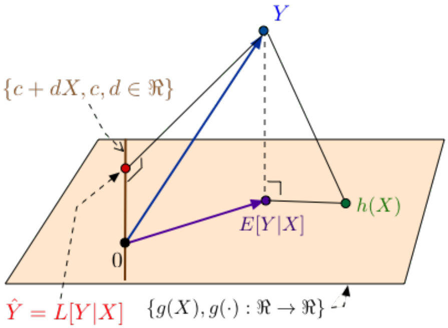

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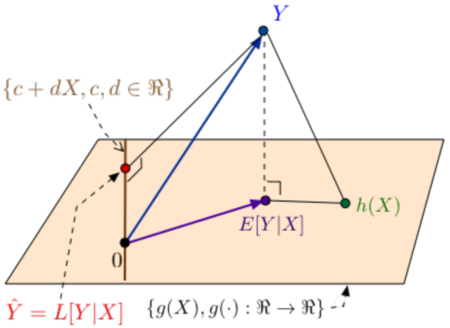

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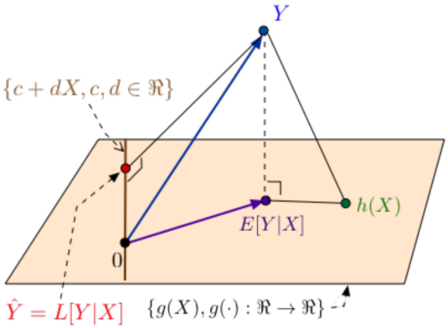

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- MMSE: $E[Y|X]$ minimizes $E[(Y - g(X))^2]$ over all $g(\cdot)$

CS70: Markov Chains.

Markov Chains 1

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1. Examples
2. Definition
3. First Passage Time

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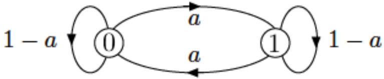

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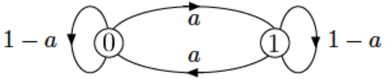

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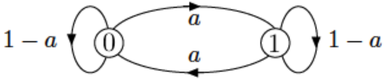

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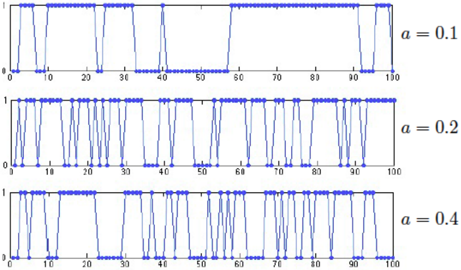

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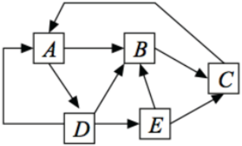
At each step, the MC follows one of the outgoing arrows of the current state, with equal probabilities.



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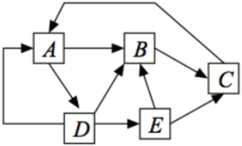

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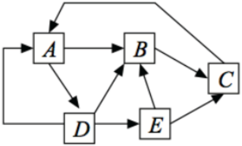

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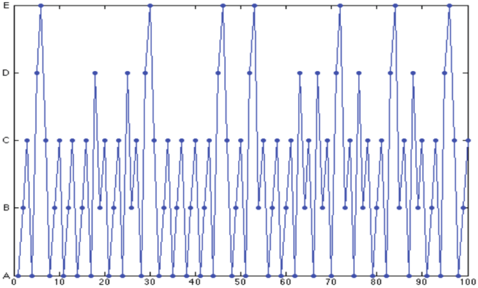

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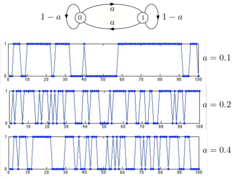

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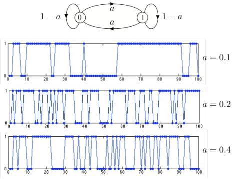

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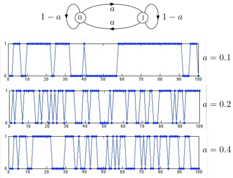

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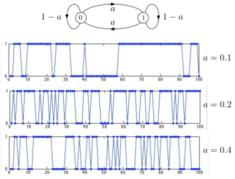

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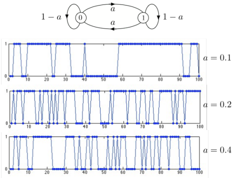

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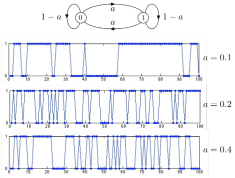

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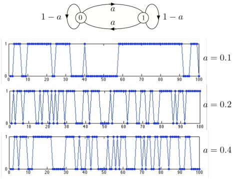

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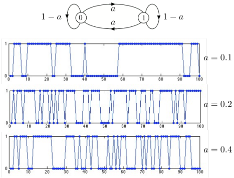

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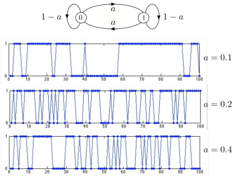

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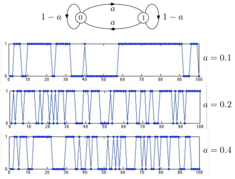

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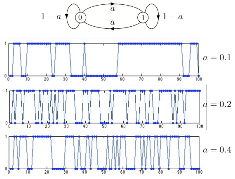

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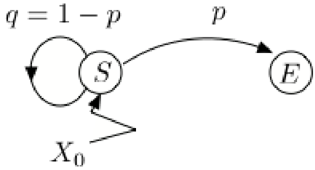

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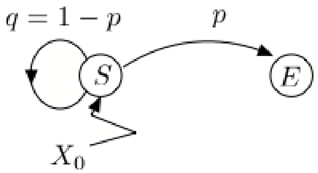

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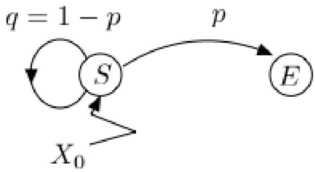

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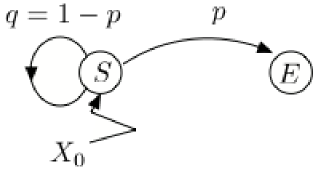

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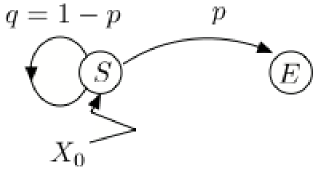

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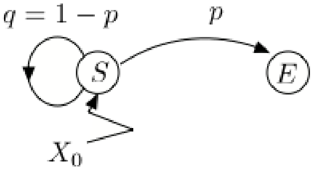

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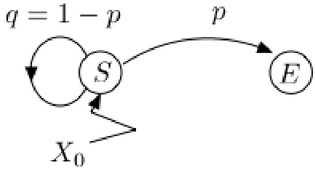

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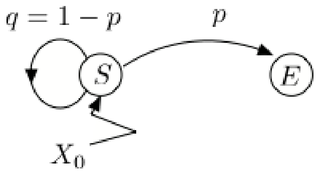

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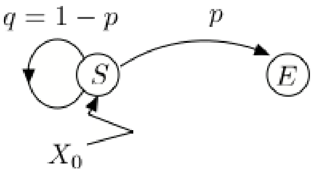

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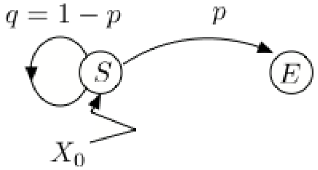

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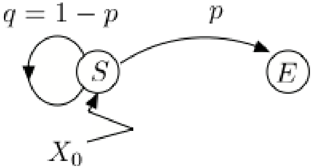

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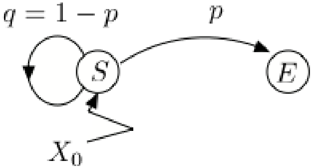

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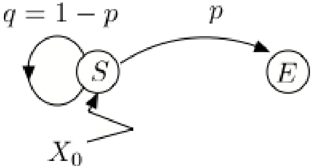

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$$\beta(S) = 1 + q\beta(S) + p0.$$

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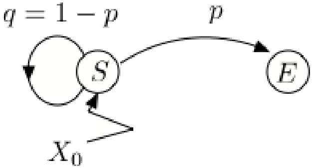

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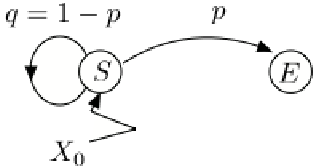

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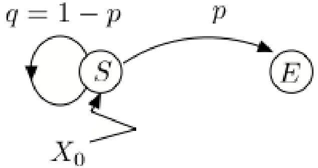

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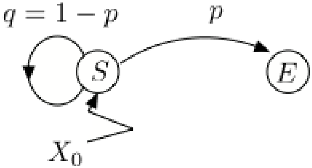

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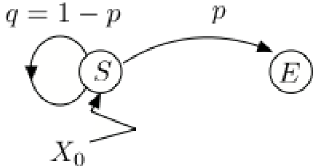

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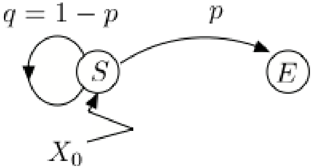

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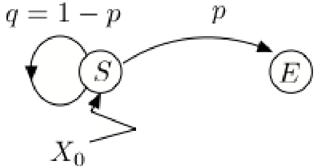

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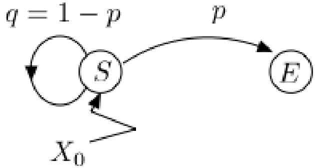

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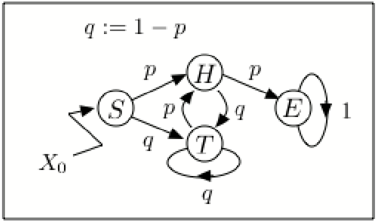

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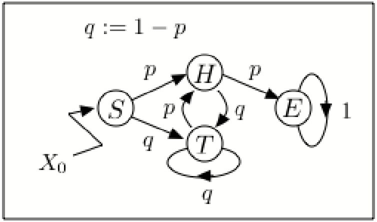

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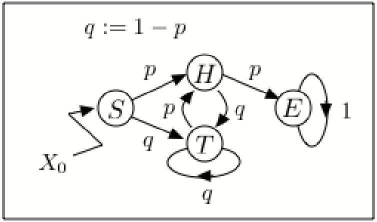

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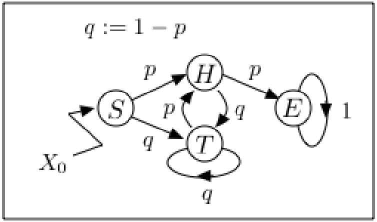

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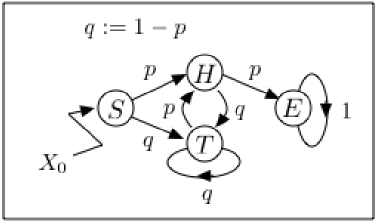

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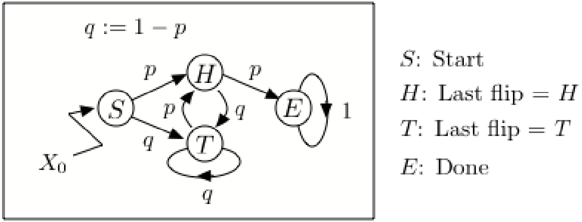

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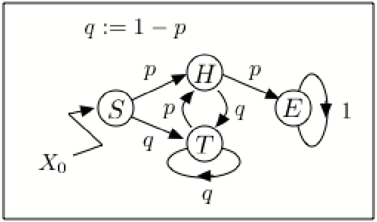

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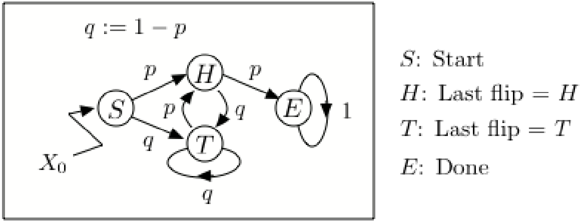

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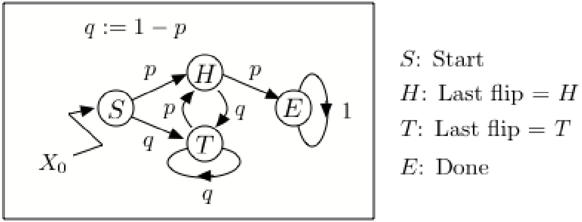

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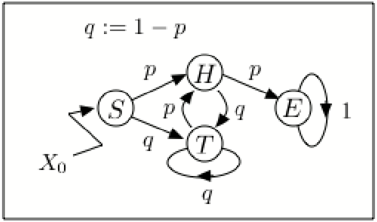

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Solving, we find $\beta(S) = 2 + 3qp^{-1} + q^2p^{-2}$. (E.g., $\beta(S) = 6$ if $p = 1/2$.)

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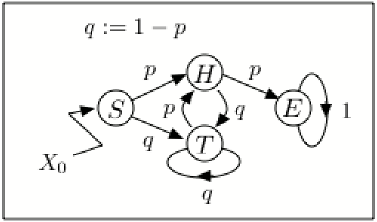

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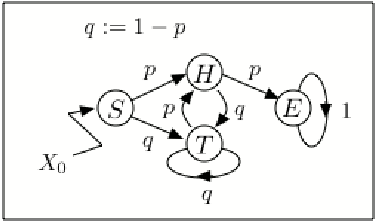
First Passage Time - Example 2



S: Start

H: Last flip = H

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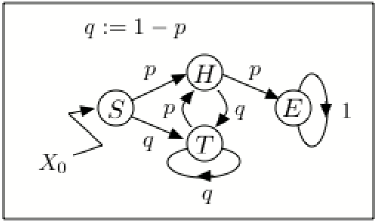
Let us justify the first step equation for $\beta(T)$.

First Passage Time - Example 2



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Let us justify the first step equation for $\beta(T)$. The others are similar.

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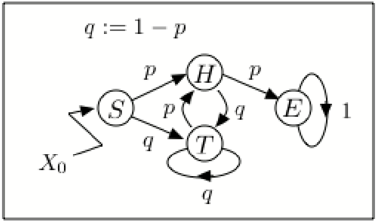

S: Start

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$N(T)$ – number of steps, starting from T until the MC hits E.

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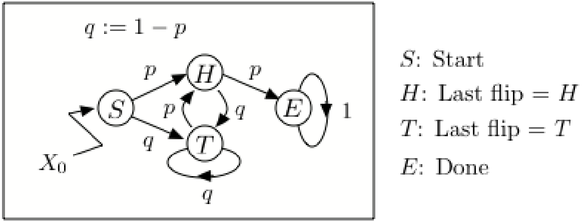

Let us justify the first step equation for $\beta(T)$. The others are similar.

$N(T)$ – number of steps, starting from T until the MC hits E.
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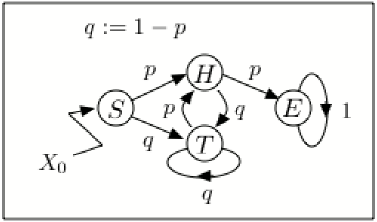

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Let us justify the first step equation for $\beta(T)$. The others are similar.

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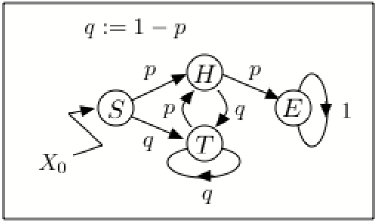

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$N(H)$ – be defined similarly.
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$$N(T) = 1 + Z \times N(H) + (1 - Z) \times N'(T)$$

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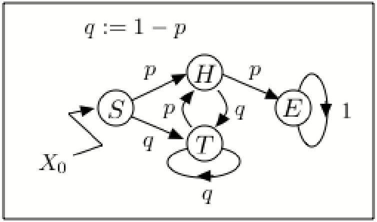

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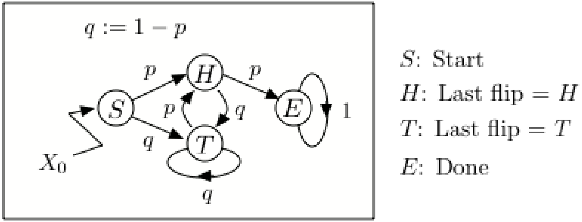

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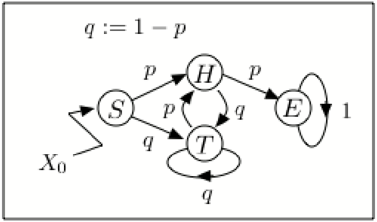

S: Start
H: Last flip $= H$
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Let us justify the first step equation for $\beta(T)$. The others are similar.

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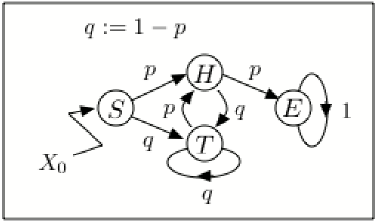

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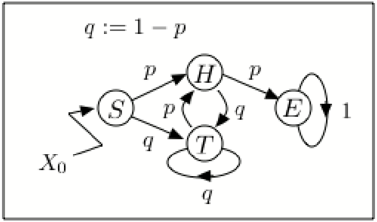
Let us justify the first step equation for $\beta(T)$. The others are similar.

$N(T)$ – number of steps, starting from T until the MC hits E.
$N(H)$ – be defined similarly.
$N'(T)$ – number of steps after the second visit to T until MC hits E.

$$N(T) = 1 + Z \times N(H) + (1 - Z) \times N'(T)$$

where $Z = 1\{\text{first flip in } T \text{ is } H\}$. Since Z and $N(H)$ are independent, and Z and $N'(T)$ are independent, taking expectations, we get

First Passage Time - Example 2



S: Start
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Let us justify the first step equation for $\beta(T)$. The others are similar.

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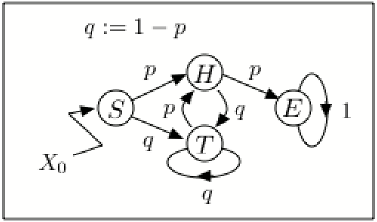

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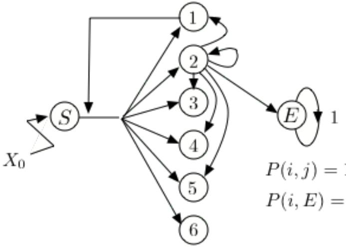

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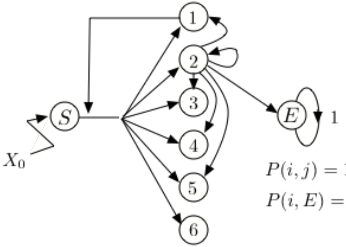

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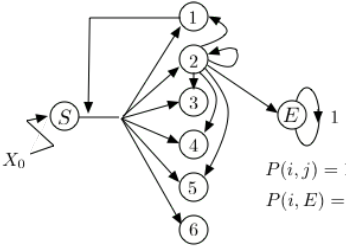

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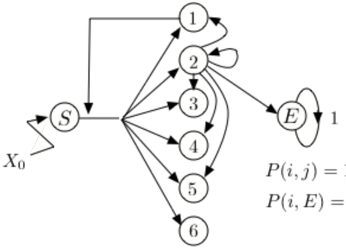

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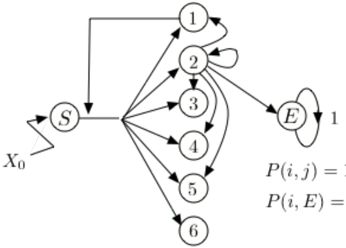

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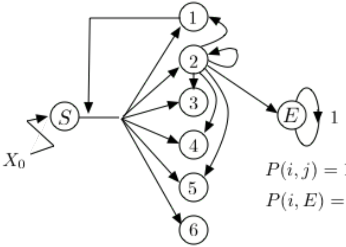

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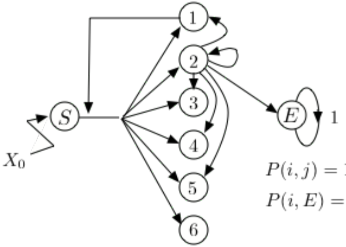

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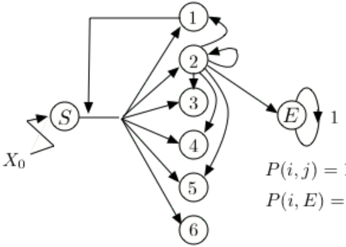

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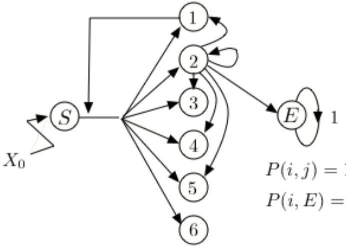

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$$\Rightarrow \cdots \beta(S) = 8.4.$$

First Passage Time - Example 4

You try to go up a ladder that has 20 rungs.

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You try to go up a ladder that has 20 rungs.
Each step, succeed or go up one rung with probability $p = 0.9$.

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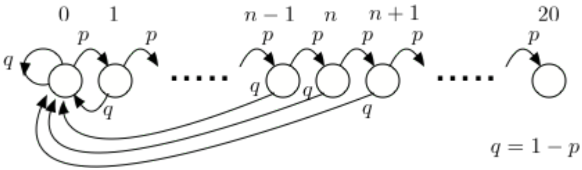

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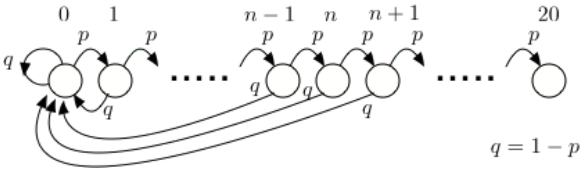

$$\beta(n) = 1 + p\beta(n+1) + q\beta(0), 0 \leq n < 19$$

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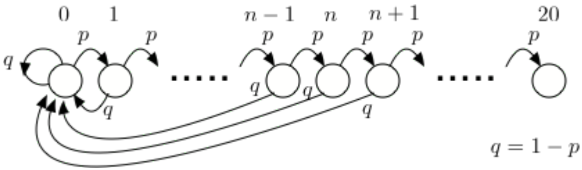

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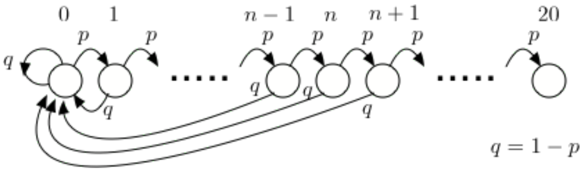

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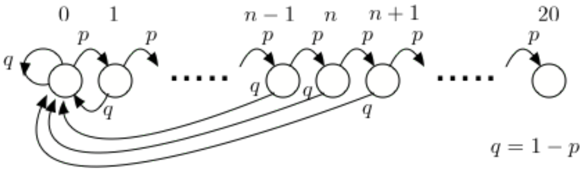

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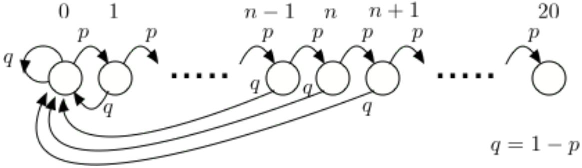

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See Lecture Note 24 for algebra.

First Passage Time - Example 5

Game of “heads or tails” using coin with ‘heads’ probability $p < 0.5$.

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Start with \$10.

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Start with \$10.
Each step, flip yields ‘heads’, earn \$1.

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What is the probability that you reach \$100 before \$0?

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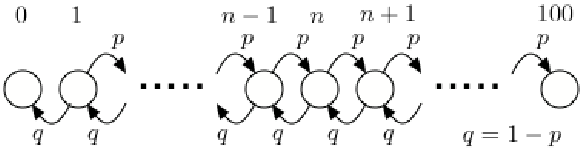

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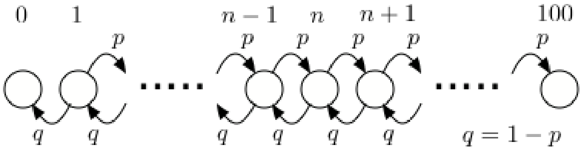

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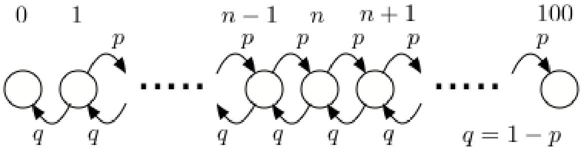

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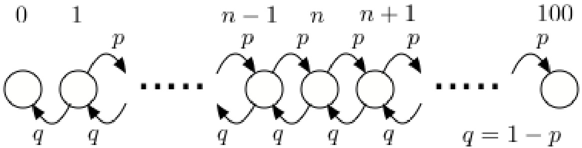

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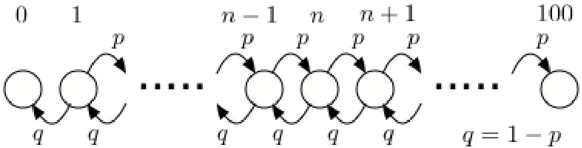

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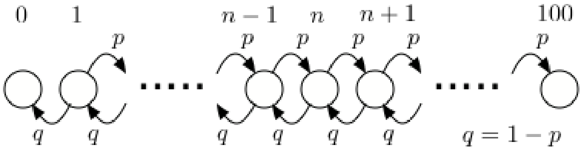

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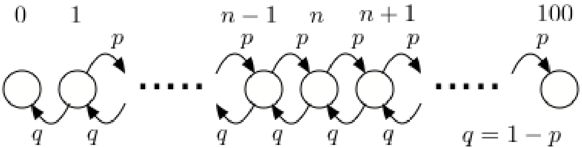

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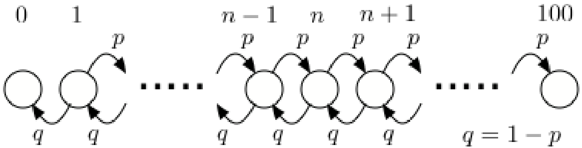

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First Passage Time - Example 5

Game of "heads or tails" using coin with 'heads' probability $p < 0.5$.
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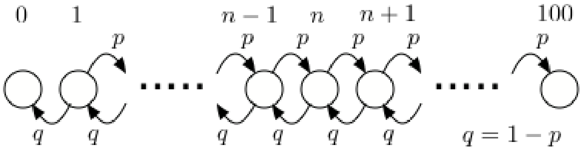

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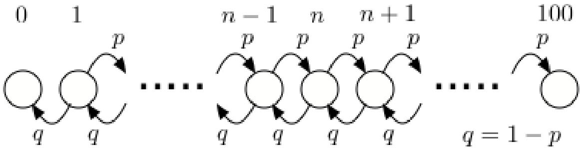

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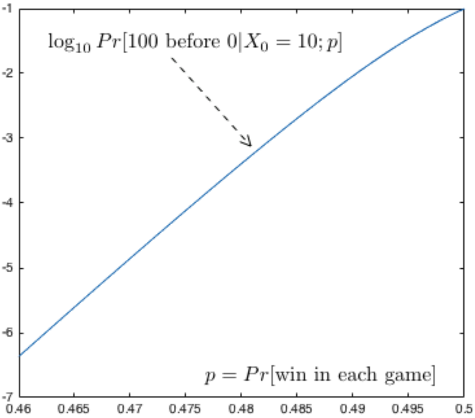

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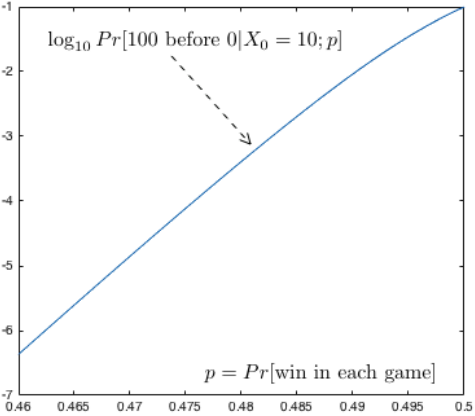

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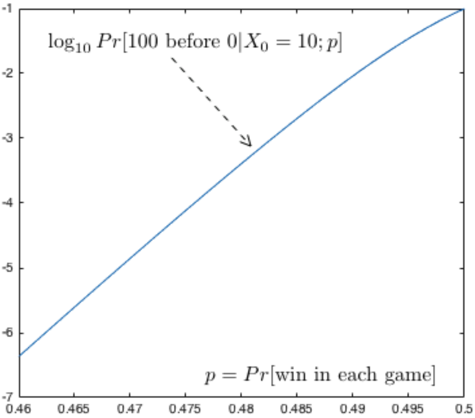

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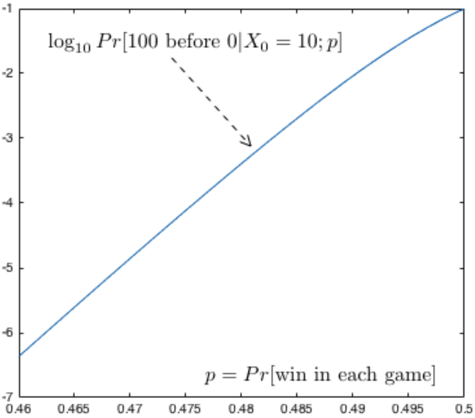

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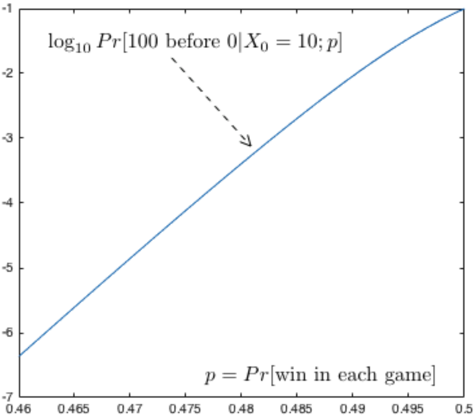

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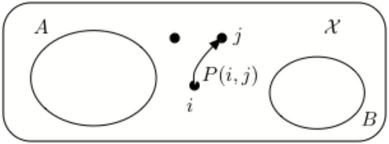

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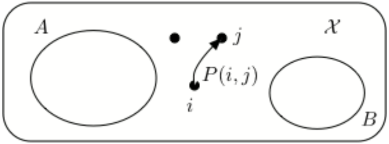

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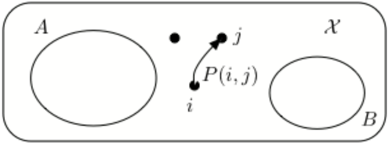

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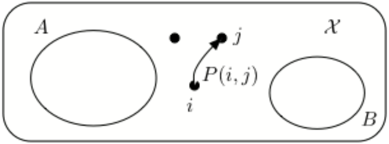

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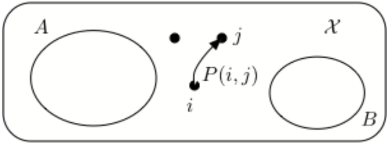

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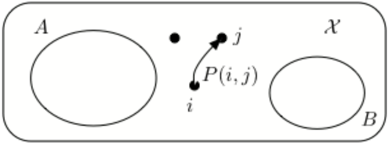

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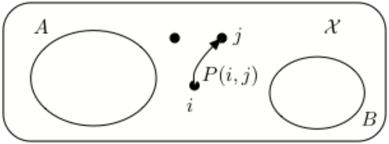

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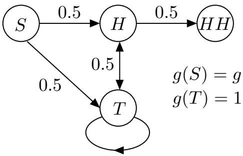

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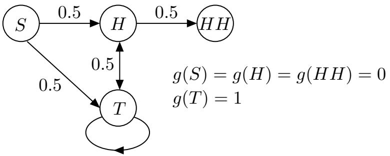

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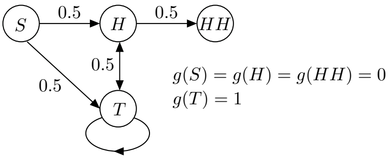

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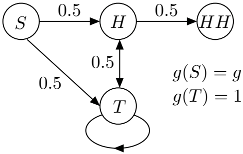

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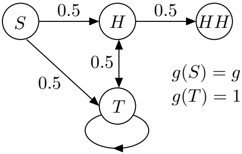

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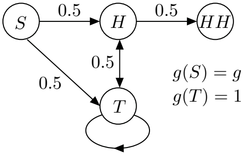

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Solving, we find $\gamma(S) = 2.5$.

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1. $Pr[X_{n+1} = j \mid X_0, \ldots, X_n = i] = P(i,j), i,j \in \mathscr{X}$
2. $T_A = \min\{n \geq 0 \mid X_n \in A\}$
3. $\alpha(i) = Pr[T_A < T_B | X_0 = i] \Rightarrow FSE$
4. $\beta(i) = E[T_A | X_0 = i] \Rightarrow FSE$
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