Today

Finish Linear Regression:

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Best linear function prediction of Y given X.

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MMSE: Best Function that predicts Y from S.

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Conditional Expectation.

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Applications to random processes.

LLSE

LLSE

Theorem

LLSE

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Consider two RVs X, Y with a given distribution $Pr[X = x, Y = y]$.
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$^{(*)}$ Recall that $cov(X,Y) = E[(X - E[X])(Y - E[Y])]$ and $var[X] = E[(X - E[X])^2]$.

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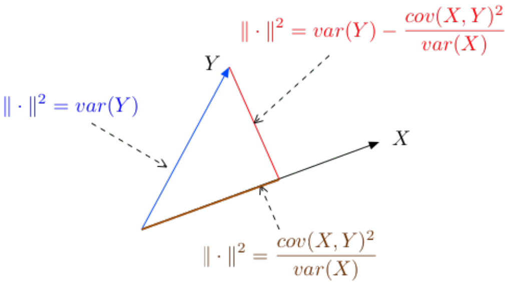

Vector Y at dimension ω is $\frac{1}{\sqrt{\Omega}} Y(\omega)$

Linear Regression Examples

Example 1:

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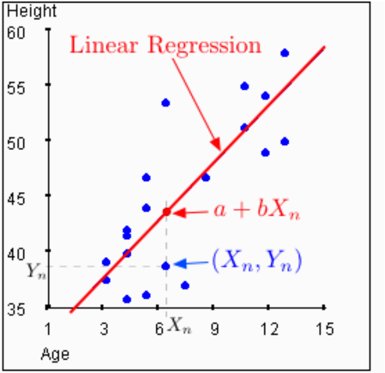

Linear Regression Examples

Example 2:

Linear Regression Examples

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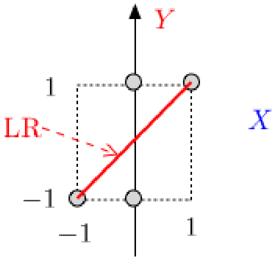

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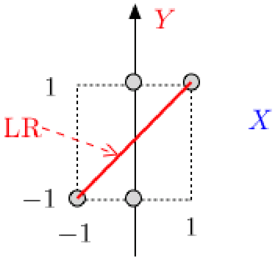

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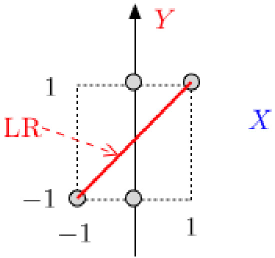

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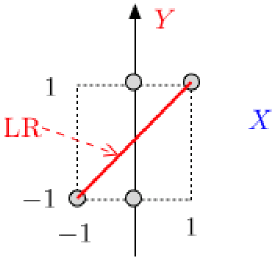

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Linear Regression Examples

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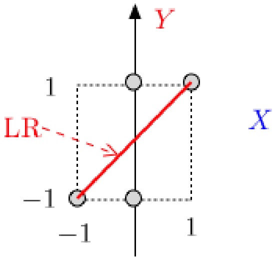

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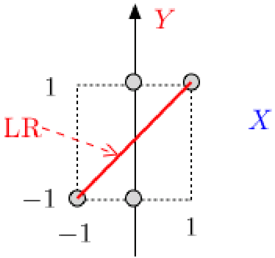

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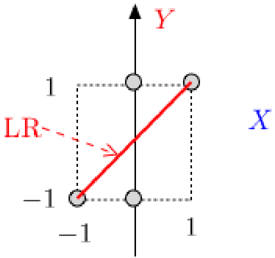

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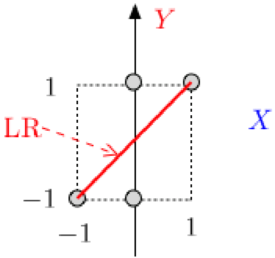

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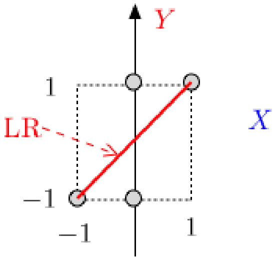

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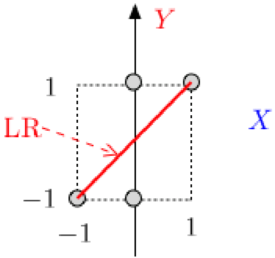

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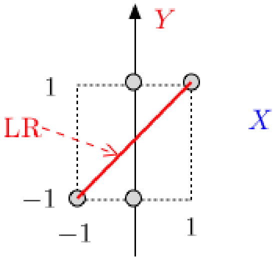

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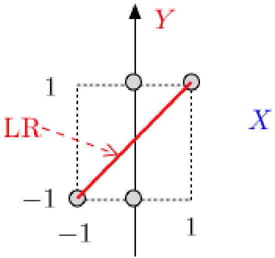

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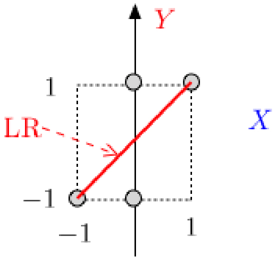

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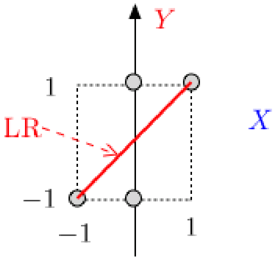

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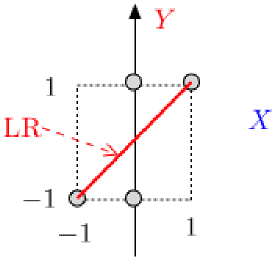

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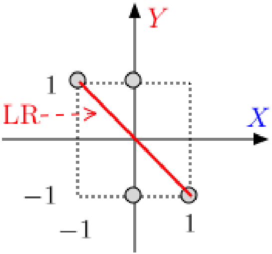

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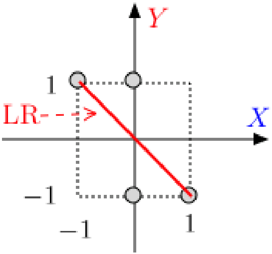

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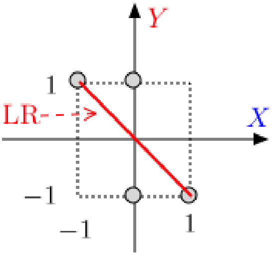

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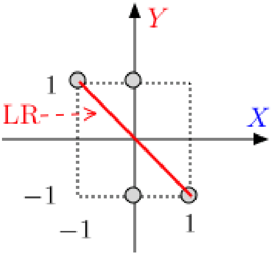

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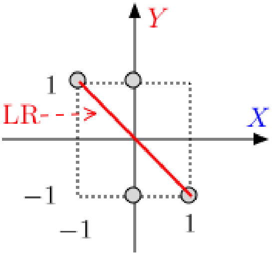

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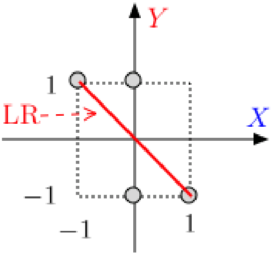

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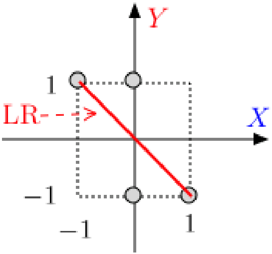

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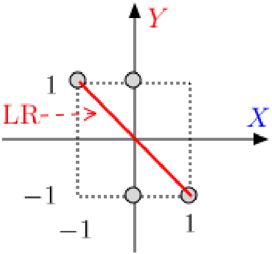

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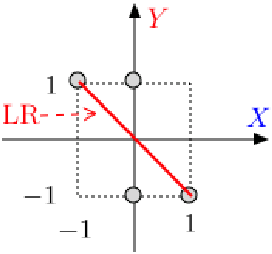

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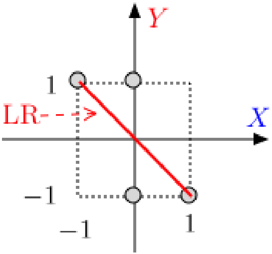

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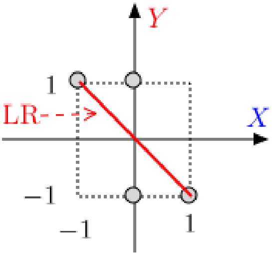

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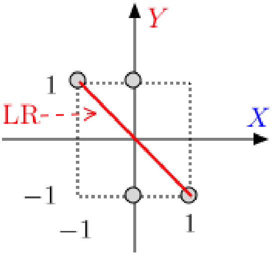

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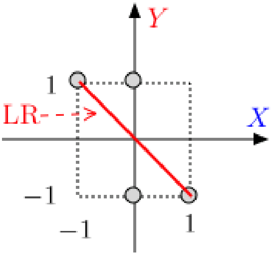

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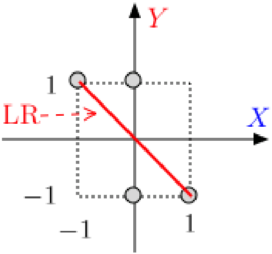

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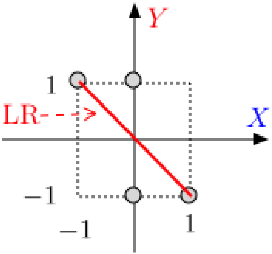

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Linear Regression Examples

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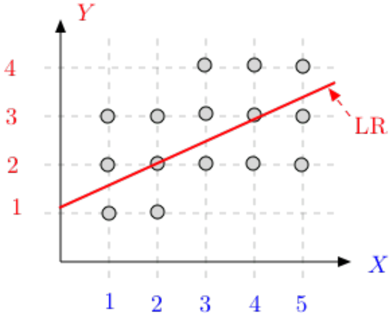

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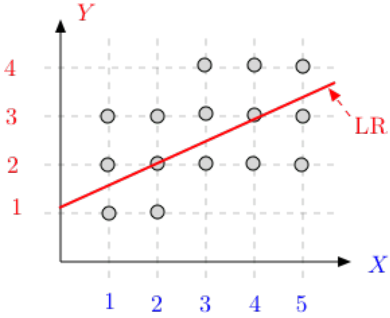

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Linear Regression Examples

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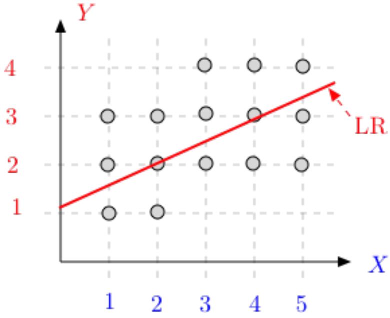

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Linear Regression Examples

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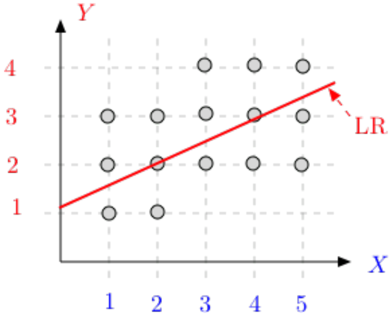

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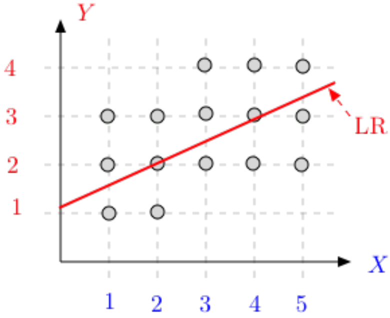

We find:

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Linear Regression Examples

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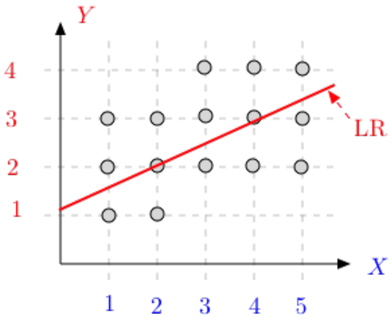

We find:

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Linear Regression Examples

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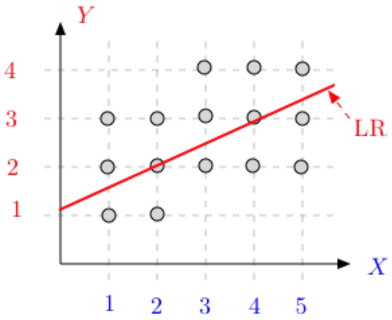

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Linear Regression Examples

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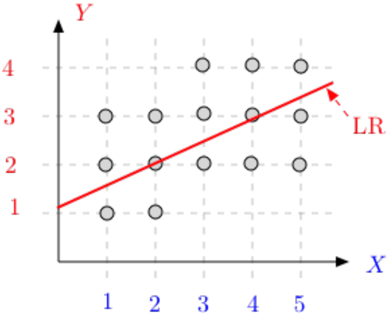

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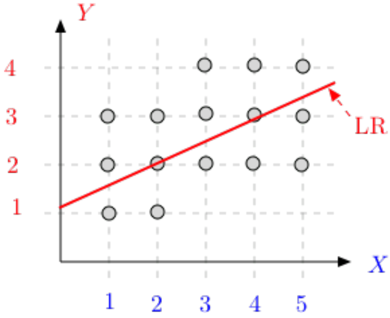

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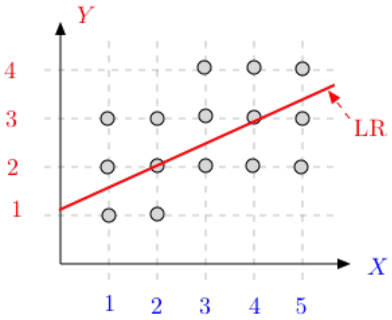

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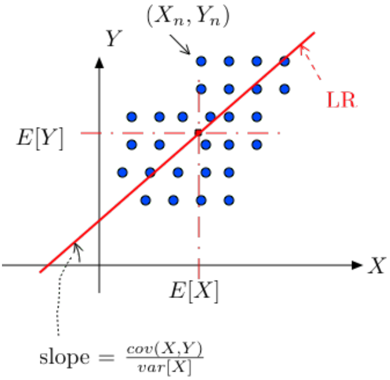
$$\text{LR: } \hat{Y} = 2.5 + \frac{0.9}{2}(X - 3) = 1.15 + 0.45X.$$

LR: Another Figure



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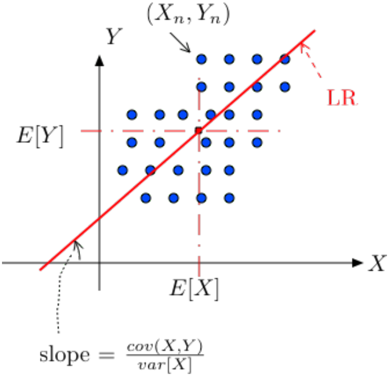

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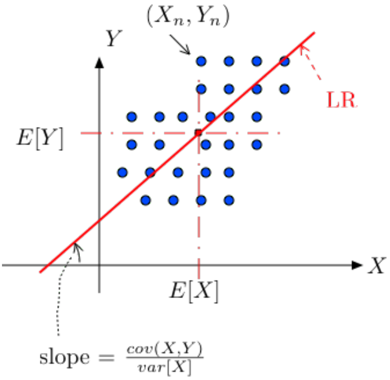

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CS70: Noninear Regression.

1. Review: joint distribution, LLSE
2. Quadratic Regression
3. Definition of Conditional expectation
4. Properties of CE
5. Applications: Diluting, Mixing, Rumors
6. CE = MMSE

Review

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Definitions Let X and Y be RVs on Ω.

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Recall the non-Bayesian and Bayesian viewpoints.

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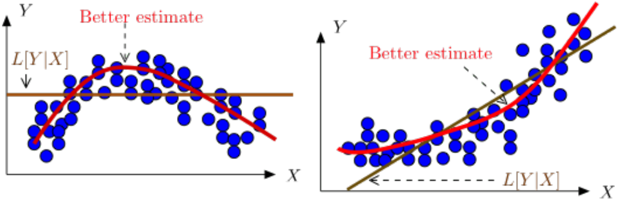

Nonlinear Regression: Motivation

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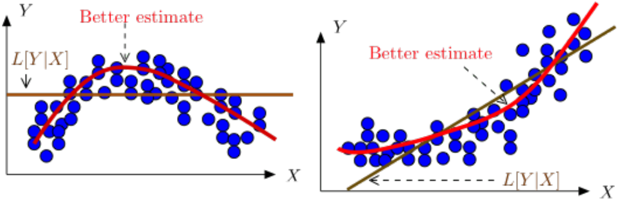

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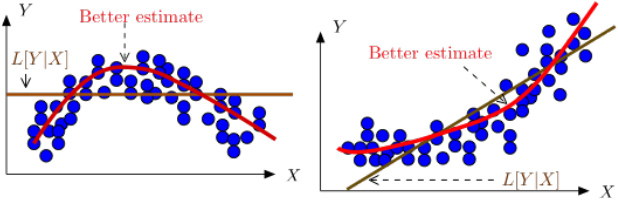

Our goal: explore estimates $\hat{Y} = g(X)$ for nonlinear functions $g(\cdot)$.

Quadratic Regression

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Proof: $E[Y|X=x] = E[Y|A]$ with $A = \{\omega : X(\omega) = x\}$. $\square$

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(e) Let $h(X) = 1$ in (d).

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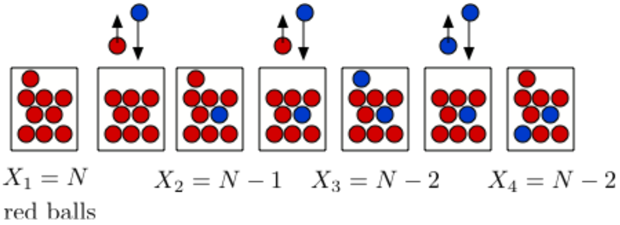

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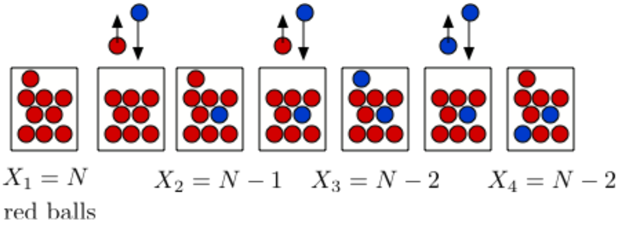

Each step, pick ball from well-mixed urn.

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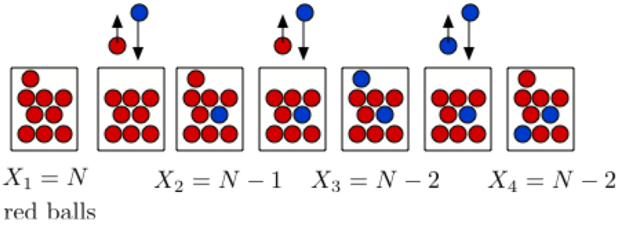

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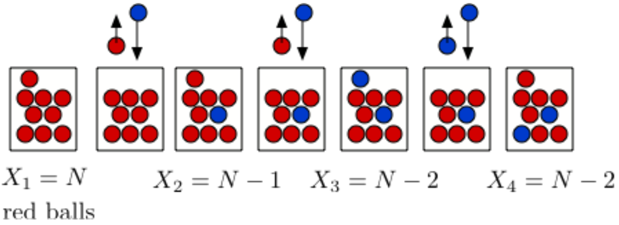

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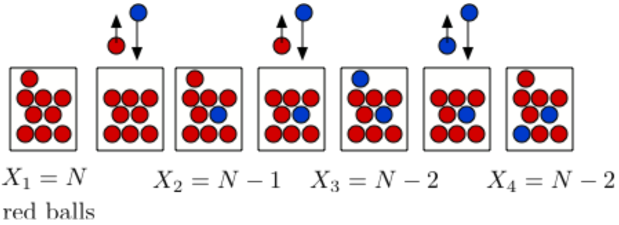
Application: Diluting



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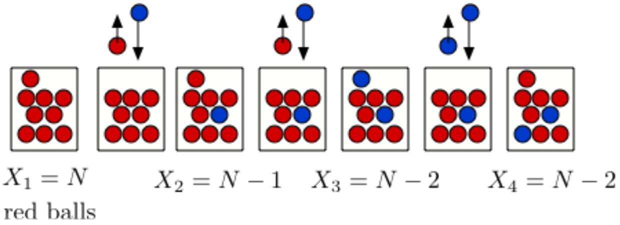

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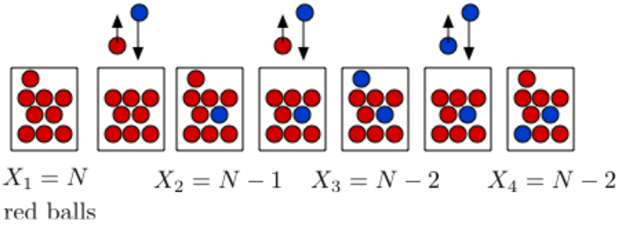

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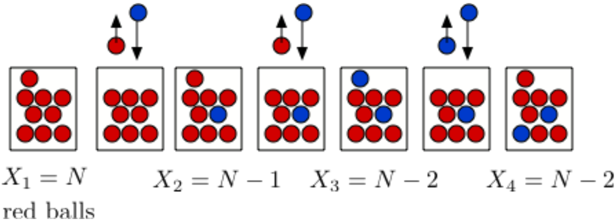

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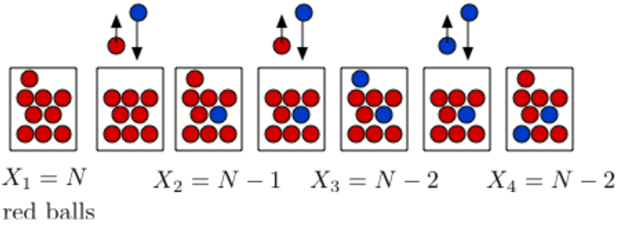

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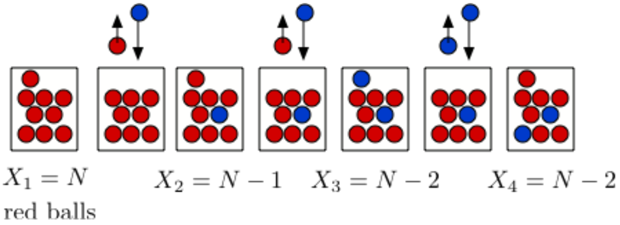

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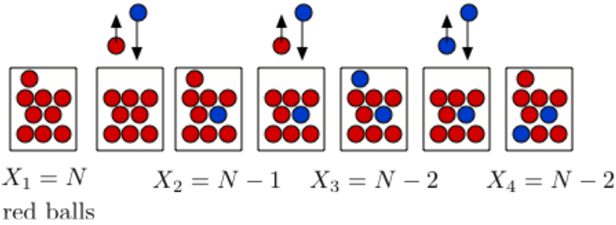

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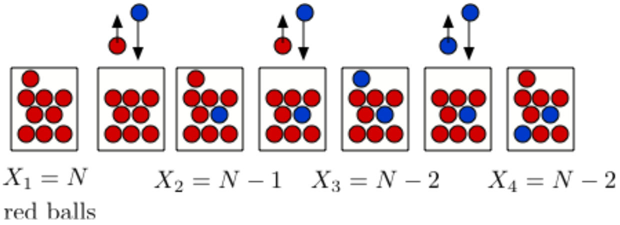

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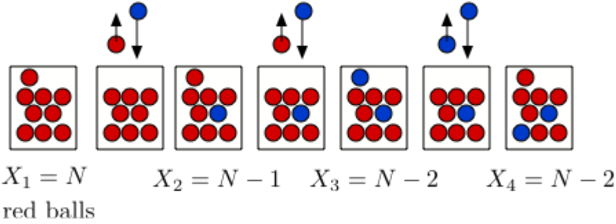

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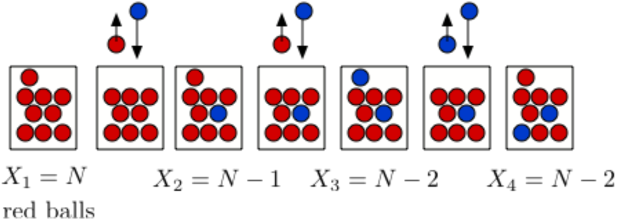

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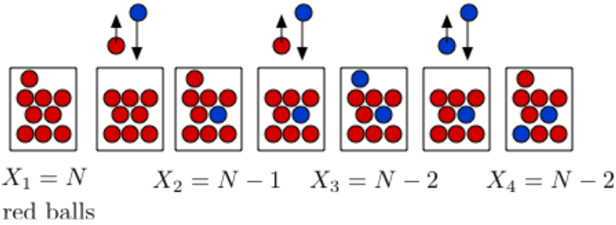

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Diluting

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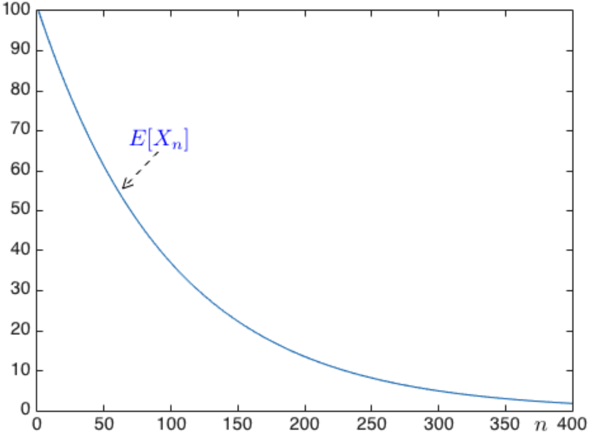

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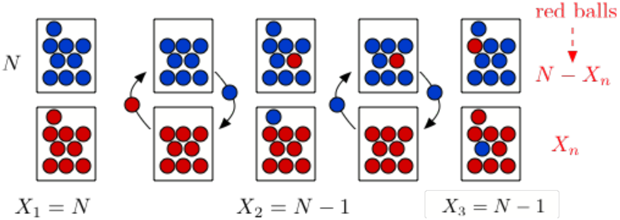
Application: Mixing



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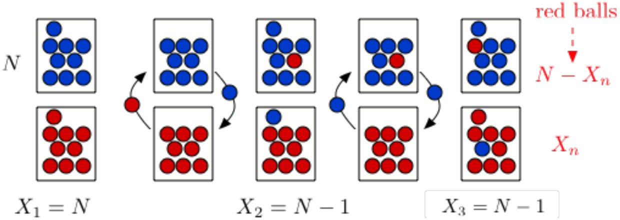

Each step, pick ball from each well-mixed urn.

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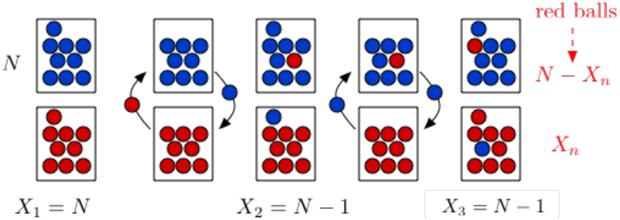

Each step, pick ball from each well-mixed urn. Transfer it to other urn.

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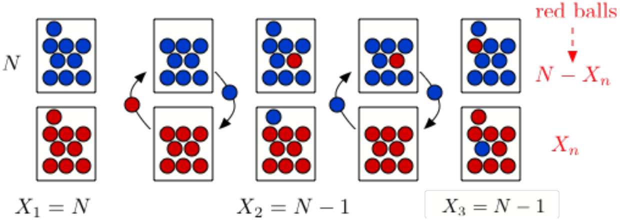

Each step, pick ball from each well-mixed urn. Transfer it to other urn.
Let X_n be the number of red balls in the bottom urn at step n.

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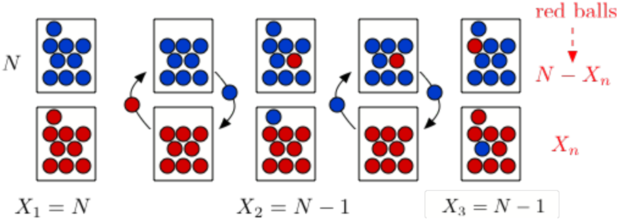

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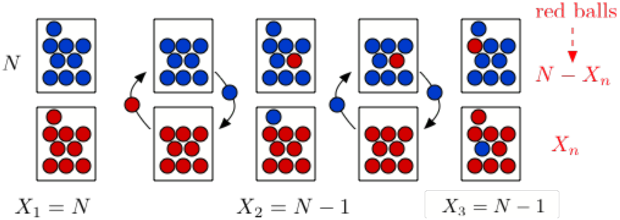

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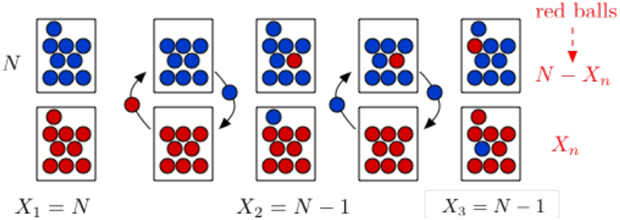

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where $p = (1 - m/N)^2$ (B goes up, R down)

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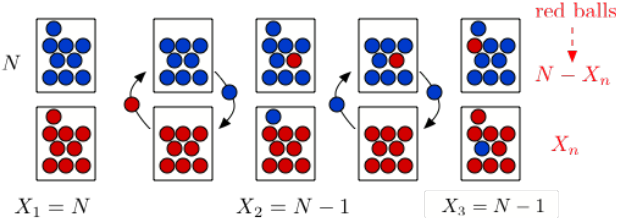

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Given $X_n = m$, $X_{n+1} = m+1$ w.p. p and $X_{n+1} = m-1$ w.p. q

where $p = (1 - m/N)^2$ (B goes up, R down)
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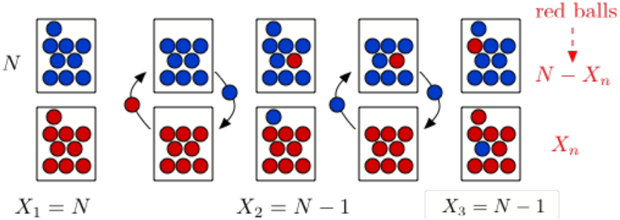
Application: Mixing



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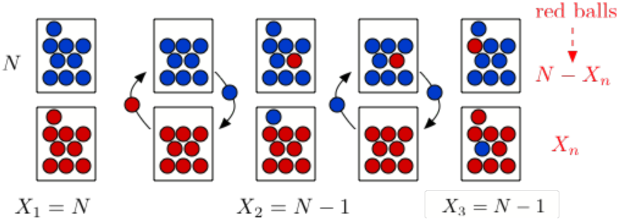

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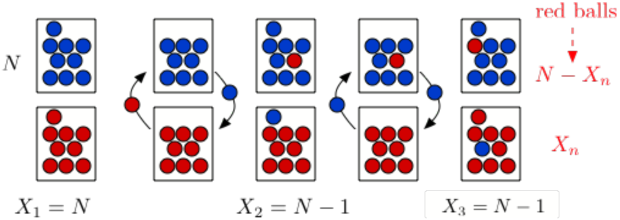

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\end{aligned}$$

Hence,

$$E[X_n] = \frac{1 - \rho^{n-1}}{1 - \rho} + \rho^{n-1} N, n \geq 1.$$

Application: Mixing

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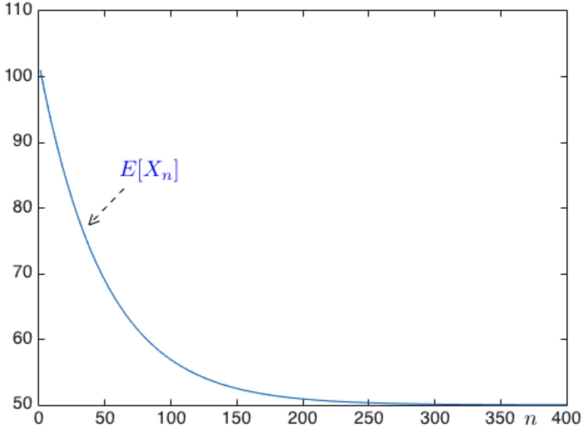

Application: Going Viral

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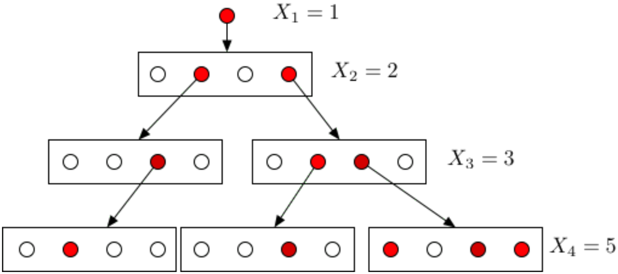

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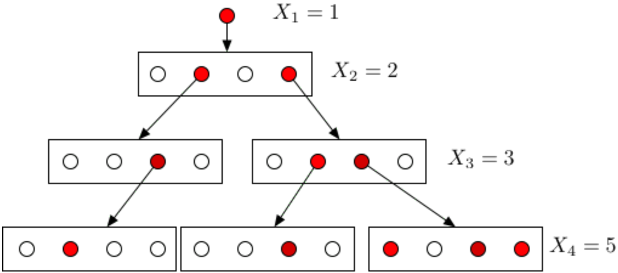
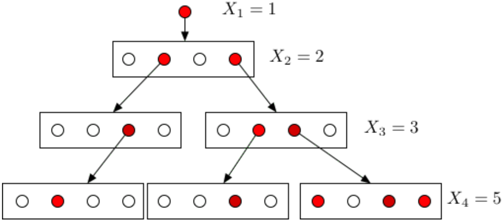


In this example, $d = 4$.

Application: Going Viral



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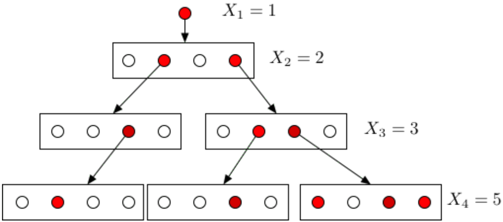
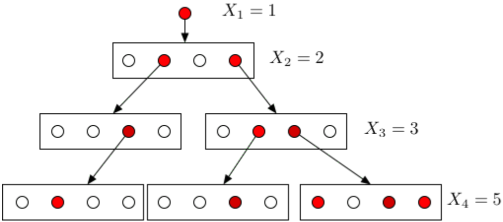


Fact:

Application: Going Viral



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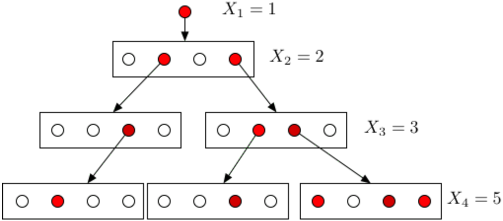
Application: Going Viral



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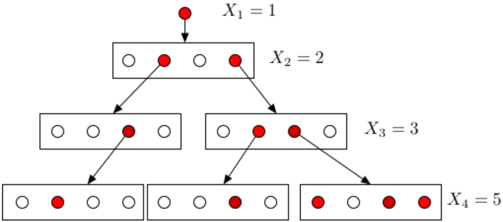
Then, $E[X] < \infty$ iff $pd < 1$.

Application: Going Viral



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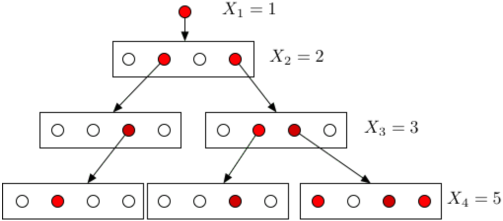
Application: Going Viral



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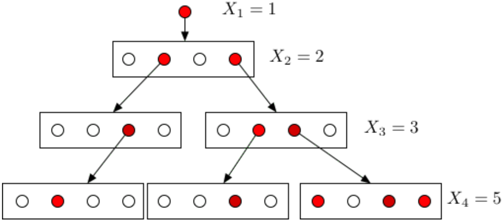

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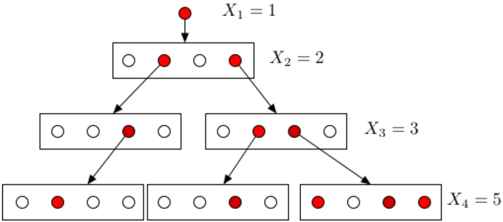

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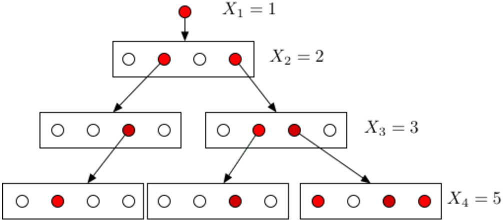

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If $pd < 1$, then $E[X_1 + \cdots + X_n] \leq (1 - pd)^{-1}$

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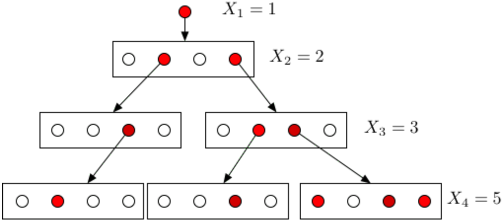

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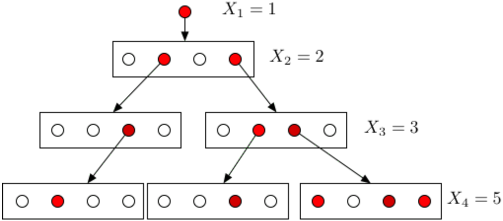

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If $pd \geq 1$, then for all C one can find n s.t.

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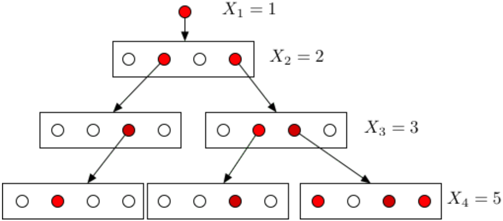
□

Application: Going Viral



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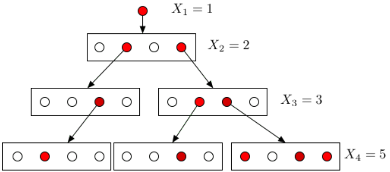
In fact, one can show that $pd \geq 1 \implies Pr[X = \infty] > 0$.

Application: Going Viral



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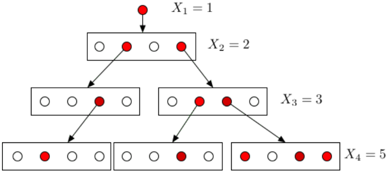

An easy extension:

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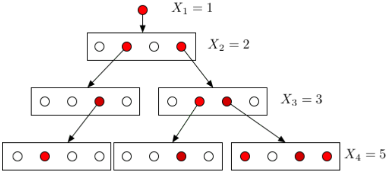

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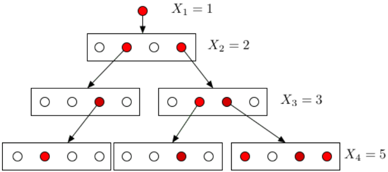

An easy extension: Assume that everyone has an independent number D_i of friends with $E[D_i] = d$. Then, the same fact holds.

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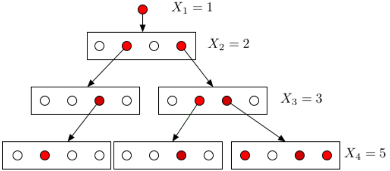

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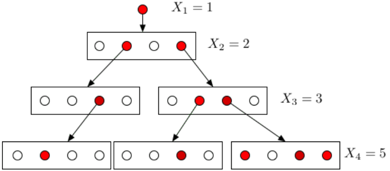

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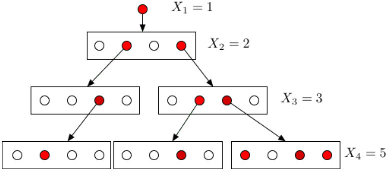

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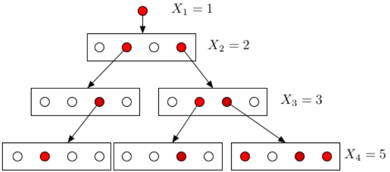

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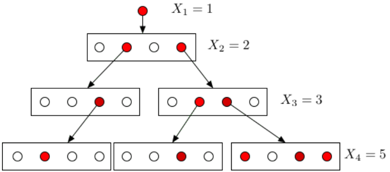

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Consequently, $E[X_{n+1} | X_n = k] = E[p(D_1 + \cdots + D_k)] = pdk$.

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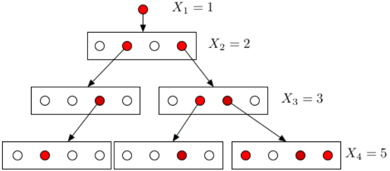

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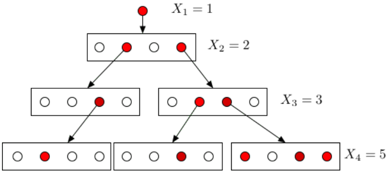

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To see this, note that given $X_n = k$, and given the numbers of friends $D_1 = d_1, \ldots, D_k = d_k$ of these X_n people, one has $X_{n+1} = B(d_1 + \cdots + d_k, p)$. Hence,

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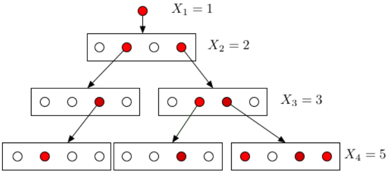
Finally, $E[X_{n+1} | X_n] = pdX_n$, and $E[X_{n+1}] = pdE[X_n]$.

Application: Going Viral



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We conclude as before.

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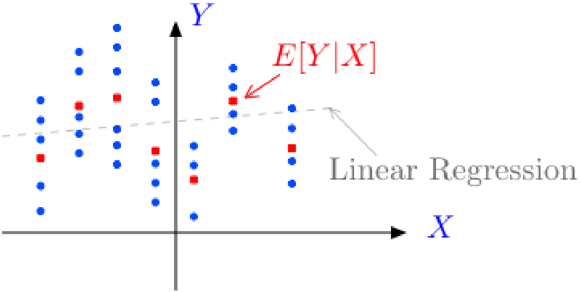

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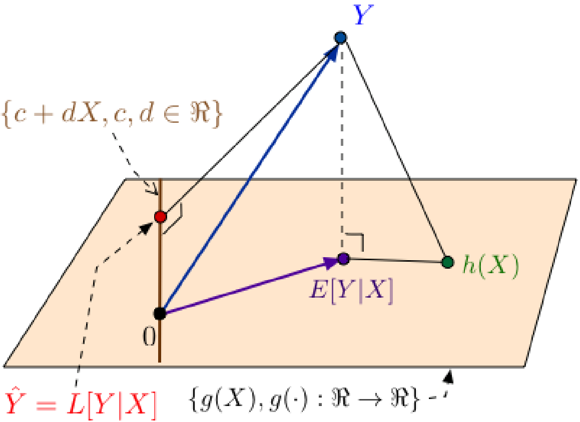
$E[Y|X]$ and $L[Y|X]$ as projections



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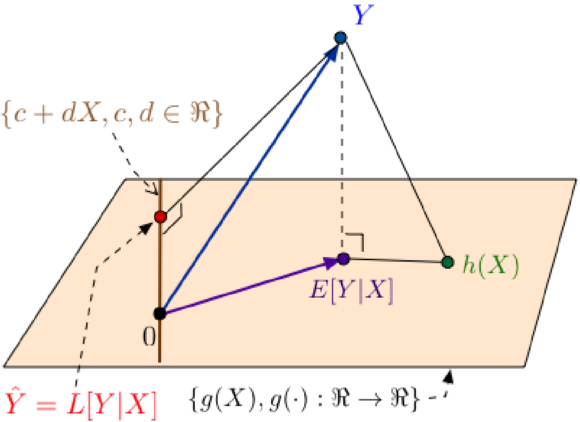

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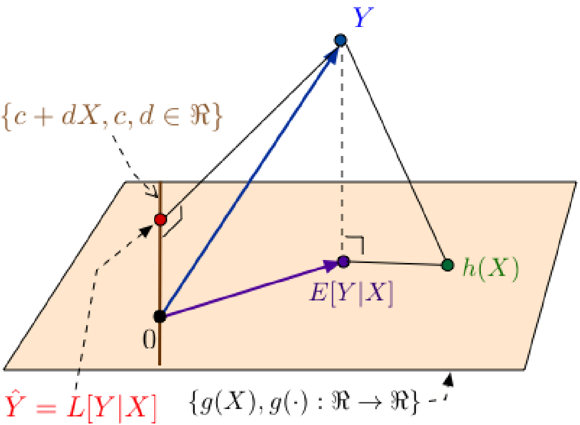

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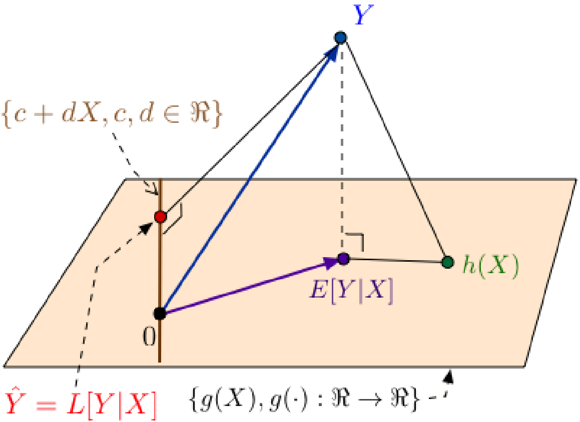

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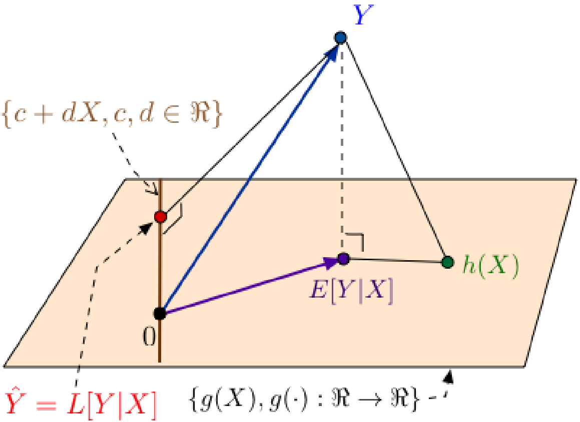

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