Today

Finish Linear Regression:
 Best linear function prediction of Y given X.

MMSE: Best Function that predicts Y from S.

Conditional Expectation.

Applications to random processes.

LLSE

Theorem
Consider two RVs X, Y with a given distribution $Pr[X = x, Y = y]$. Then,

$$L[Y|X] = \hat{Y} = E[Y] + \frac{cov(X,Y)}{var(X)}(X - E[X]).$$

Proof 1:
$Y - \hat{Y} = (Y - E[Y]) - \frac{cov(X,Y)}{var[X]}(X - E[X])$. $E[Y - \hat{Y}] = 0$ by linearity.

Also, $E[(Y - \hat{Y})X] = 0$, after a bit of algebra. (See next slide.)

Combine brown inequalities: $E[(Y - \hat{Y})(c + dX)] = 0$ for any c, d.
Since: $\hat{Y} = \alpha + \beta X$ for some α, β, so $\exists c, d$ s.t. $\hat{Y} - a - bX = c + dX$.
Then, $E[(Y - \hat{Y})(\hat{Y} - a - bX)] = 0, \forall a, b$. Now,

$$\begin{aligned} E[(Y - a - bX)^2] &= E[(Y - \hat{Y} + \hat{Y} - a - bX)^2] \\ &= E[(Y - \hat{Y})^2] + E[(\hat{Y} - a - bX)^2] + 0 \geq E[(Y - \hat{Y})^2]. \end{aligned}$$

This shows that $E[(Y - \hat{Y})^2] \leq E[(Y - a - bX)^2]$, for all (a, b).
Thus $\hat{Y}$ is the LLSE. $\square$

A Bit of Algebra

$Y - \hat{Y} = (Y - E[Y]) - \frac{cov(X,Y)}{var[X]}(X - E[X]).$

Hence, $E[Y - \hat{Y}] = 0$. We want to show that $E[(Y - \hat{Y})X] = 0$.

Note that

$$E[(Y - \hat{Y})X] = E[(Y - \hat{Y})(X - E[X])],$$

because $E[(Y - \hat{Y})E[X]] = 0$.

Now,

$$
\begin{aligned}
& E[(Y - \hat{Y})(X - E[X])] \\
& \quad = E[(Y - E[Y])(X - E[X])] - \frac{cov(X,Y)}{var[X]} E[(X - E[X])(X - E[X])] \\
& \quad =^{(*)} cov(X,Y) - \frac{cov(X,Y)}{var[X]} var[X] = 0. \quad \square
\end{aligned}
$$

$^{(*)}$ Recall that $cov(X,Y) = E[(X - E[X])(Y - E[Y])]$ and $var[X] = E[(X - E[X])^2]$.

Estimation Error

We saw that the LLSE of Y given X is

$$L[Y|X] = \hat{Y} = E[Y] + \frac{cov(X,Y)}{var(X)}(X - E[X]).$$

How good is this estimator?
Or what is the mean squared estimation error?

We find

$$\begin{aligned}
E[|Y - L[Y|X]|^2] &= E[(Y - E[Y] - (cov(X,Y)/var(X))(X - E[X]))^2] \\
&= E[(Y - E[Y])^2] - 2(cov(X,Y)/var(X))E[(Y - E[Y])(X - E[X])] \\
&\quad + (cov(X,Y)/var(X))^2 E[(X - E[X])^2] \\
&= var(Y) - \frac{cov(X,Y)^2}{var(X)}.
\end{aligned}$$

Without observations, the estimate is $E[Y]$. The error is $var(Y)$. Observing X reduces the error.

Estimation Error: A Picture

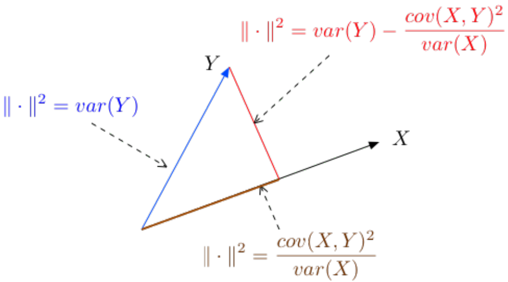
We saw that

$$L[Y|X] = \hat{Y} = E[Y] + \frac{cov(X,Y)}{var(X)}(X - E[X])$$

and

$$E[|Y - L[Y|X]|^2] = var(Y) - \frac{cov(X,Y)^2}{var(X)}.$$

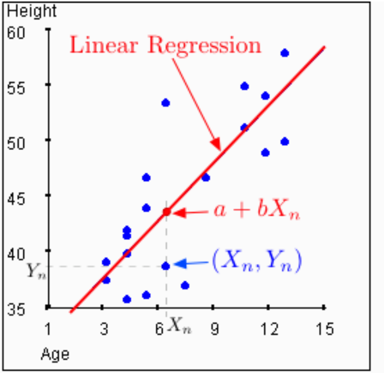
Here is a picture when $E[X] = 0, E[Y] = 0$:
Dimensions correspond to sample points, uniform sample space.



Vector Y at dimension ω is $\frac{1}{\sqrt{\Omega}} Y(\omega)$

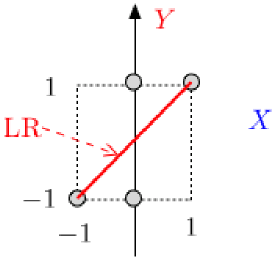
Linear Regression Examples

Example 1:



Linear Regression Examples

Example 2:



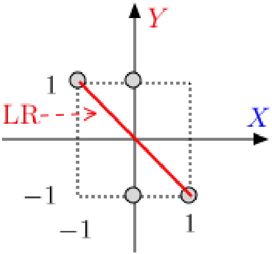
We find:

$$E[X] = 0; E[Y] = 0; E[X^2] = 1/2; E[XY] = 1/2;$$
$$var[X] = E[X^2] - E[X]^2 = 1/2; cov(X,Y) = E[XY] - E[X]E[Y] = 1/2;$$

LR: $\hat{Y} = E[Y] + \frac{cov(X,Y)}{var[X]}(X - E[X]) = X.$

Linear Regression Examples

Example 3:

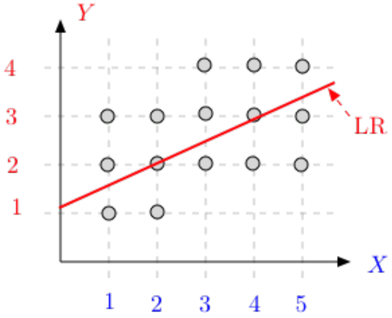


We find:

$$E[X] = 0; E[Y] = 0; E[X^2] = 1/2; E[XY] = -1/2;$$
$$var[X] = E[X^2] - E[X]^2 = 1/2; cov(X, Y) = E[XY] - E[X]E[Y] = -1/2;$$
$$\text{LR: } \hat{Y} = E[Y] + \frac{cov(X, Y)}{var[X]}(X - E[X]) = -X.$$

Linear Regression Examples

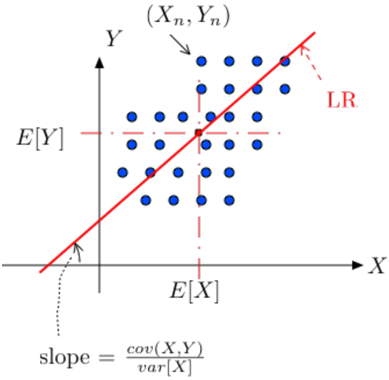
Example 4:



We find:

$$E[X] = 3; E[Y] = 2.5; E[X^2] = (3/15)(1 + 2^2 + 3^2 + 4^2 + 5^2) = 11;$$
$$E[XY] = (1/15)(1 \times 1 + 1 \times 2 + \cdots + 5 \times 4) = 8.4;$$
$$var[X] = 11 - 9 = 2; cov(X, Y) = 8.4 - 3 \times 2.5 = 0.9;$$
$$\text{LR: } \hat{Y} = 2.5 + \frac{0.9}{2}(X - 3) = 1.15 + 0.45X.$$

LR: Another Figure



Note that

- the LR line goes through $(E[X], E[Y])$
- its slope is $\frac{cov(X,Y)}{var(X)}$.

Summary

$\boxed{\text{Linear Regression}}$

1. Linear Regression: $L[Y|X] = E[Y] + \frac{cov(X,Y)}{var(X)}(X - E[X])$
2. Non-Bayesian: minimize $\sum_n (Y_n - a - bX_n)^2$
3. Bayesian: minimize $E[(Y - a - bX)^2]$

CS70: Noninear Regression.

1. Review: joint distribution, LLSE
2. Quadratic Regression
3. Definition of Conditional expectation
4. Properties of CE
5. Applications: Diluting, Mixing, Rumors
6. CE = MMSE

Review

Definitions Let X and Y be RVs on Ω.

- Joint Distribution: $Pr[X = x, Y = y]$
- Marginal Distribution: $Pr[X = x] = \sum_y Pr[X = x, Y = y]$
- Conditional Distribution: $Pr[Y = y | X = x] = \frac{Pr[X=x,Y=y]}{Pr[X=x]}$
- LLSE: $L[Y|X] = a + bX$ where a, b minimize $E[(Y - a - bX)^2]$.

We saw that

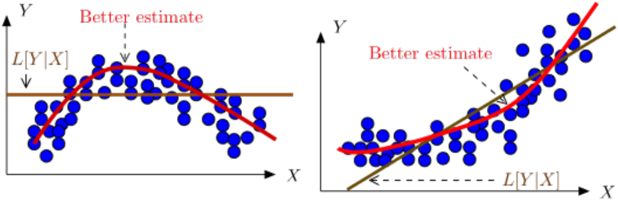
$$L[Y|X] = E[Y] + \frac{cov(X,Y)}{var[X]}(X - E[X]).$$

Recall the non-Bayesian and Bayesian viewpoints.

Nonlinear Regression: Motivation

There are many situations where a good guess about Y given X is not linear.

E.g., (diameter of object, weight), (school years, income), (PSA level, cancer risk).



Our goal: explore estimates $\hat{Y} = g(X)$ for nonlinear functions $g(\cdot)$.

Quadratic Regression

Let X, Y be two random variables defined on the same probability space.

Definition: The quadratic regression of Y over X is the random variable

$$Q[Y|X] = a + bX + cX^2$$

where a, b, c are chosen to minimize $E[(Y - a - bX - cX^2)^2]$.

Derivation: We set to zero the derivatives w.r.t. a, b, c. We get

$$\begin{aligned}
0 &= E[Y - a - bX - cX^2] \\
0 &= E[(Y - a - bX - cX^2)X] \\
0 &= E[(Y - a - bX - cX^2)X^2]
\end{aligned}$$

We solve these three equations in the three unknowns (a, b, c).

Note: These equations imply that $E[(Y - Q[Y|X])h(X)] = 0$ for any $h(X) = d + eX + fX^2$. That is, the estimation error is orthogonal to all the quadratic functions of X. Hence, $Q[Y|X]$ is the projection of Y onto the space of quadratic functions of X.

Conditional Expectation

Definition Let X and Y be RVs on Ω. The conditional expectation of Y given X is defined as

$$E[Y|X] = g(X)$$

where

$$g(x) := E[Y|X = x] := \sum_{y} y Pr[Y = y|X = x].$$

Fact

$$E[Y|X = x] = \sum_{\omega} Y(\omega) Pr[\omega|X = x].$$

Proof: $E[Y|X = x] = E[Y|A]$ with $A = \{\omega : X(\omega) = x\}$. $\square$

Deja vu, all over again?

Have we seen this before? Yes.

Is anything new? Yes.

The idea of defining $g(x) = E[Y|X = x]$ and then $E[Y|X] = g(X)$.

Big deal? Quite! Simple but most convenient.

Recall that $L[Y|X] = a + bX$ is a function of X.

This is similar: $E[Y|X] = g(X)$ for some function $g(\cdot)$.

In general, $g(X)$ is not linear, i.e., not $a + bX$. It could be that $g(X) = a + bX + cX^2$. Or that $g(X) = 2\sin(4X) + \exp\{-3X\}$. Or something else.

Properties of CE

$$E[Y|X=x] = \sum_y y Pr[Y=y|X=x]$$

Theorem

(a) X, Y independent $\Rightarrow E[Y|X] = E[Y]$;

(b) $E[aY + bZ|X] = aE[Y|X] + bE[Z|X]$;

(c) $E[Yh(X)|X] = h(X)E[Y|X], \forall h(\cdot)$;

(d) $E[h(X)E[Y|X]] = E[h(X)Y], \forall h(\cdot)$;

(e) $E[E[Y|X]] = E[Y]$.

Proof:

(a),(b) Obvious

(c) $$E[Yh(X)|X=x] = \sum_\omega Y(\omega)h(X(\omega))Pr[\omega|X=x]$$
$$= \sum_\omega Y(\omega)h(x)Pr[\omega|X=x] = h(x)E[Y|X=x]$$

Properties of CE

$$E[Y|X=x] = \sum_y yPr[Y=y|X=x]$$

Theorem

(a) X, Y independent $\Rightarrow E[Y|X] = E[Y]$;
(b) $E[aY + bZ|X] = aE[Y|X] + bE[Z|X]$;
(c) $E[Yh(X)|X] = h(X)E[Y|X], \forall h(\cdot)$;
(d) $E[h(X)E[Y|X]] = E[h(X)Y], \forall h(\cdot)$;
(e) $E[E[Y|X]] = E[Y]$.

Proof: (continued)

$$\begin{aligned}
\text{(d)}\quad E[h(X)E[Y|X]] &= \sum_x h(x) E[Y|X=x] Pr[X=x] \\
&= \sum_x h(x) \sum_y yPr[Y=y|X=x] Pr[X=x] \\
&= \sum_x h(x) \sum_y yPr[X=x, y=y] \\
&= \sum_{x,y} h(x) yPr[X=x, y=y] = E[h(X)Y].
\end{aligned}$$

$\square$

Properties of CE

$$E[Y|X=x] = \sum_y yPr[Y=y|X=x]$$

Theorem
(a) X, Y independent $\Rightarrow E[Y|X] = E[Y]$;
(b) $E[aY + bZ|X] = aE[Y|X] + bE[Z|X]$;
(c) $E[Yh(X)|X] = h(X)E[Y|X], \forall h(\cdot)$;
(d) $E[h(X)E[Y|X]] = E[h(X)Y], \forall h(\cdot)$;
(e) $E[E[Y|X]] = E[Y]$.

Proof: (continued)

(e) Let $h(X) = 1$ in (d).

$\square$

Properties of CE

Theorem
(a) X, Y independent $\Rightarrow E[Y|X] = E[Y]$;
(b) $E[aY + bZ|X] = aE[Y|X] + bE[Z|X]$;
(c) $E[Yh(X)|X] = h(X)E[Y|X], \forall h(\cdot)$;
(d) $E[h(X)E[Y|X]] = E[h(X)Y], \forall h(\cdot)$;
(e) $E[E[Y|X]] = E[Y]$.

Note that (d) says that

$$E[(Y - E[Y|X])h(X)] = 0.$$

We say that the estimation error $Y - E[Y|X]$ is orthogonal to every function $h(X)$ of X.

We call this the projection property. More about this later.

Application: Calculating $E[Y|X]$

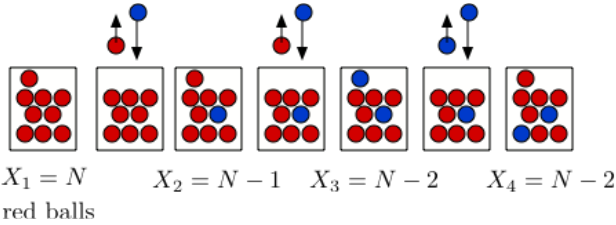
Let X, Y, Z be i.i.d. with mean 0 and variance 1. We want to calculate

$$E[2 + 5X + 7XY + 11X^2 + 13X^3 Z^2 | X].$$

We find

$$\begin{aligned}
&E[2 + 5X + 7XY + 11X^2 + 13X^3 Z^2 | X] \\
&\quad = 2 + 5X + 7XE[Y|X] + 11X^2 + 13X^3 E[Z^2|X] \\
&\quad = 2 + 5X + 7XE[Y] + 11X^2 + 13X^3 E[Z^2] \\
&\quad = 2 + 5X + 11X^2 + 13X^3 (var[Z] + E[Z]^2) \\
&\quad = 2 + 5X + 11X^2 + 13X^3.
\end{aligned}$$

Application: Diluting



Each step, pick ball from well-mixed urn. Replace with blue ball.
Let X_n be the number of red balls in the urn at step n.
What is $E[X_n]$?

Given $X_n = m$, $X_{n+1} = m - 1$ w.p. m/N (if you pick a red ball) and $X_{n+1} = m$ otherwise. Hence,

$$E[X_{n+1}|X_n = m] = m - (m/N) = m(N-1)/N = X_n \rho,$$

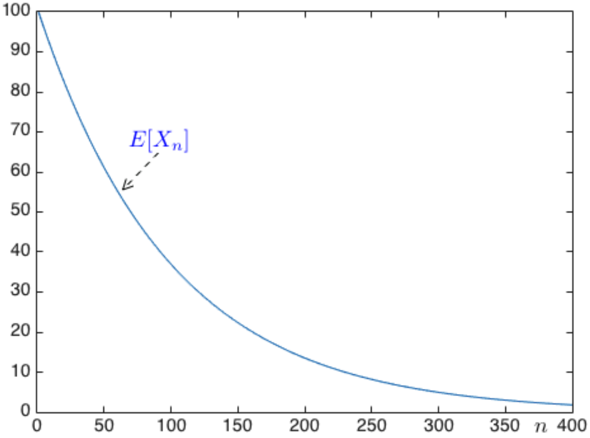
with $\rho := (N-1)/N$. Consequently,

$$E[X_{n+1}] = E[E[X_{n+1}|X_n]] = \rho E[X_n], n \geq 1.$$

$$\implies E[X_n] = \rho^{n-1} E[X_1] = N(\frac{N-1}{N})^{n-1}, n \geq 1.$$

Diluting

Here is a plot:



Diluting

By analyzing $E[X_{n+1}|X_n]$, we found that $E[X_n] = N(\frac{N-1}{N})^{n-1}, n \geq 1$.

Here is another argument for that result.

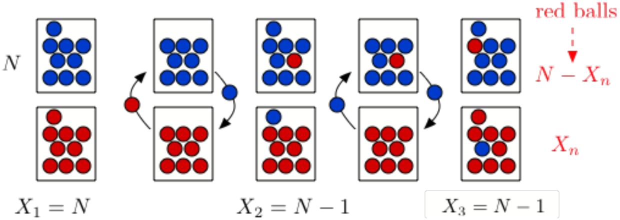
Consider one particular red ball, say ball k.
Each step, it remains red w.p. $(N-1)/N$, if different ball picked. $\Longrightarrow$ the probability still red at step n is $[(N-1)/N]^{n-1}$. Define:

$$Y_n(k) = 1\{\text{ball } k \text{ is red at step } n\}.$$

Then, $X_n = Y_n(1) + \cdots + Y_n(N)$. Hence,

$$\begin{aligned} E[X_n] &= E[Y_n(1) + \cdots + Y_n(N)] = NE[Y_n(1)] \\ &= NPr[Y_n(1) = 1] = N[(N-1)/N]^{n-1}. \end{aligned}$$

Application: Mixing



Each step, pick ball from each well-mixed urn. Transfer it to other urn.
Let X_n be the number of red balls in the bottom urn at step n.
What is $E[X_n]$?

Given $X_n = m$, $X_{n+1} = m + 1$ w.p. p and $X_{n+1} = m - 1$ w.p. q

where $p = (1 - m/N)^2$ (B goes up, R down)
and $q = (m/N)^2$ (R goes up, B down).

Thus,
$E[X_{n+1}|X_n] = X_n + p - q = X_n + 1 - 2X_n/N = 1 + \rho X_n, \ \rho := (1 - 2/N)$.

Mixing

We saw that $E[X_{n+1}|X_n] = 1 + \rho X_n,\ \rho := (1 - 2/N)$.
Does that make sense?
Hence,

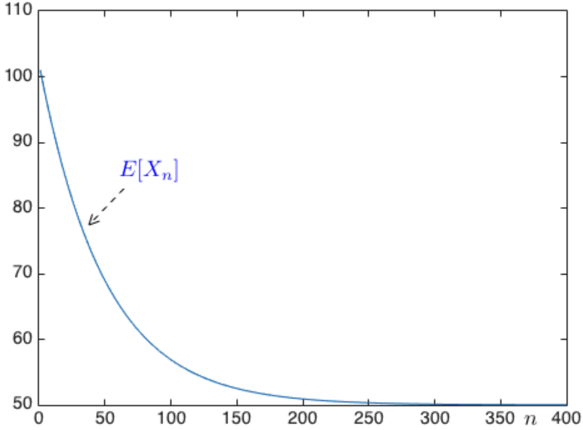
$$\begin{aligned}
&E[X_{n+1}] = 1 + \rho E[X_n] \\
&E[X_2] = 1 + \rho N; E[X_3] = 1 + \rho(1 + \rho N) = 1 + \rho + \rho^2 N \\
&E[X_4] = 1 + \rho(1 + \rho + \rho^2 N) = 1 + \rho + \rho^2 + \rho^3 N \\
&E[X_n] = 1 + \rho + \cdots + \rho^{n-2} + \rho^{n-1} N.
\end{aligned}$$

Hence,

$$E[X_n] = \frac{1 - \rho^{n-1}}{1 - \rho} + \rho^{n-1} N, n \geq 1.$$

Application: Mixing

Here is the plot.



Application: Going Viral

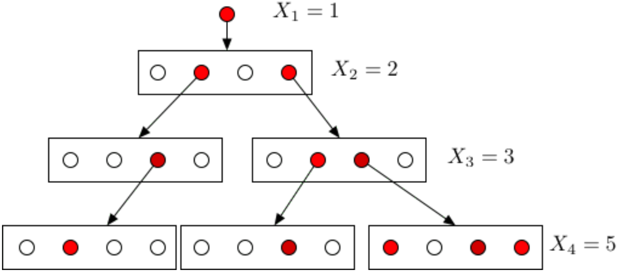
Consider a social network (e.g., Twitter).

You start a rumor (e.g., Rao is bad at making copies).

You have d friends. Each of your friend retweets w.p. p.

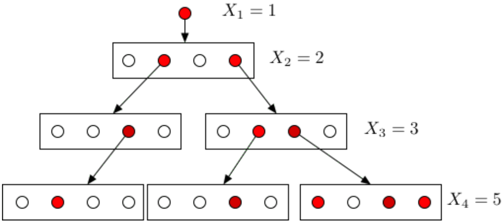
Each of your friends has d friends, etc.

Does the rumor spread? Does it die out (mercifully)?



In this example, $d = 4$.

Application: Going Viral



Fact: Number of tweets $X = \sum_{n=1}^{\infty} X_n$ where X_n is tweets in level n.
Then, $E[X] < \infty$ iff $pd < 1$.

Proof:
Given $X_n = k$, $X_{n+1} = B(kd, p)$. Hence, $E[X_{n+1}|X_n = k] = kpd$.

Thus, $E[X_{n+1}|X_n] = pdX_n$. Consequently, $E[X_n] = (pd)^{n-1}, n \geq 1$.

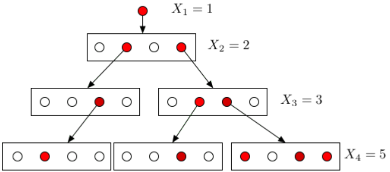
If $pd < 1$, then $E[X_1 + \cdots + X_n] \leq (1 - pd)^{-1} \implies E[X] \leq (1 - pd)^{-1}$.

If $pd \geq 1$, then for all C one can find n s.t.

$$E[X] \geq E[X_1 + \cdots + X_n] \geq C.$$ $\square$

In fact, one can show that $pd \geq 1 \implies Pr[X = \infty] > 0$.

Application: Going Viral



An easy extension: Assume that everyone has an independent number D_i of friends with $E[D_i] = d$. Then, the same fact holds.

To see this, note that given $X_n = k$, and given the numbers of friends $D_1 = d_1, \ldots, D_k = d_k$ of these X_n people, one has $X_{n+1} = B(d_1 + \cdots + d_k, p)$. Hence,

$$E[X_{n+1} | X_n = k, D_1 = d_1, \ldots, D_k = d_k] = p(d_1 + \cdots + d_k).$$

Thus, $E[X_{n+1} | X_n = k, D_1, \ldots, D_k] = p(D_1 + \cdots + D_k)$.

Consequently, $E[X_{n+1} | X_n = k] = E[p(D_1 + \cdots + D_k)] = pdk$.

Finally, $E[X_{n+1} | X_n] = pdX_n$, and $E[X_{n+1}] = pdE[X_n]$.

We conclude as before.

Application: Wald's Identity

Here is an extension of an identity we used in the last slide.

Theorem Wald's Identity

Assume that $X_1, X_2, \ldots$ and Z are independent, where

Z takes values in $\{0, 1, 2, \ldots\}$

and $E[X_n] = \mu$ for all $n \geq 1$.

Then,

$$E[X_1 + \cdots + X_Z] = \mu E[Z].$$

Proof:

$E[X_1 + \cdots + X_Z | Z = k] = \mu k$.

Thus, $E[X_1 + \cdots + X_Z | Z] = \mu Z$.

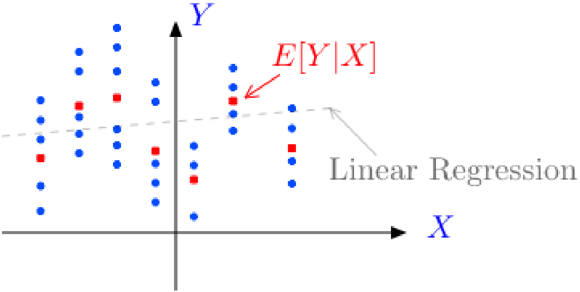
Hence, $E[X_1 + \cdots + X_Z] = E[\mu Z] = \mu E[Z]$. $\square$

CE = MMSE

Theorem
$E[Y|X]$ is the 'best' guess about Y based on X.

Specifically, it is the function $g(X)$ of X that

$$\text{minimizes } E[(Y - g(X))^2].$$



CE = MMSE

Theorem CE = MMSE

$g(X) := E[Y|X]$ is the function of X that minimizes $E[(Y - g(X))^2]$.
Proof:
Let $h(X)$ be any function of X. Then

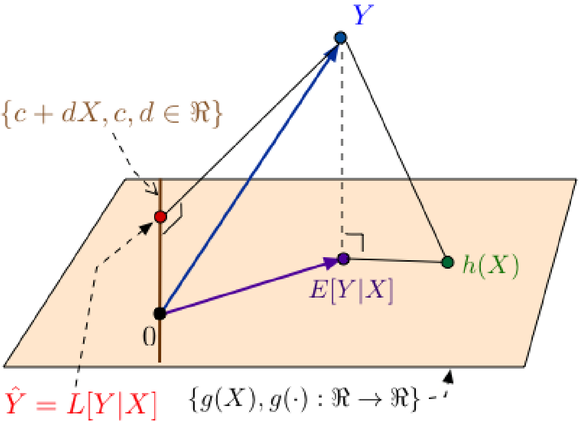
$$\begin{aligned} E[(Y-h(X))^2] &= E[(Y-g(X)+g(X)-h(X))^2] \\ &= E[(Y-g(X))^2] + E[(g(X)-h(X))^2] \\ &\quad +2E[(Y-g(X))(g(X)-h(X))]. \end{aligned}$$

But,

$E[(Y-g(X))(g(X)-h(X))] = 0$ by the projection property.

Thus, $E[(Y-h(X))^2] \geq E[(Y-g(X))^2]$. $\square$

$E[Y|X]$ and $L[Y|X]$ as projections



$L[Y|X]$ is the projection of Y on $\{a + bX, a, b \in \Re\}$: LLSE

$E[Y|X]$ is the projection of Y on $\{g(X), g(\cdot) : \Re \to \Re\}$: MMSE.

Summary

$\boxed{\text{Conditional Expectation}}$

- Definition: $E[Y|X] := \sum_y yPr[Y = y|X = x]$
- Properties: Linearity, $Y - E[Y|X] \perp h(X)$; $E[E[Y|X]] = E[Y]$
- Some Applications:
  - Calculating $E[Y|X]$
  - Diluting
  - Mixing
  - Rumors
  - Wald
- MMSE: $E[Y|X]$ minimizes $E[(Y - g(X))^2]$ over all $g(\cdot)$