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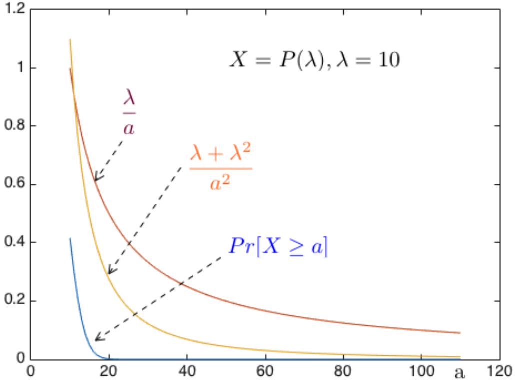

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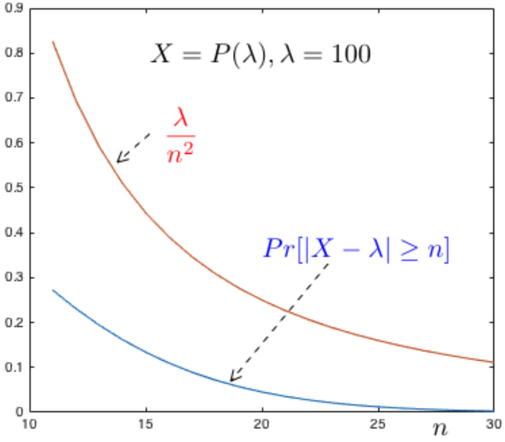

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Hence, for $a > \lambda$, $Pr[X \geq a] \leq Pr[|X - \lambda| \geq a - \lambda] \leq \frac{\lambda}{(a-\lambda)^2}$.

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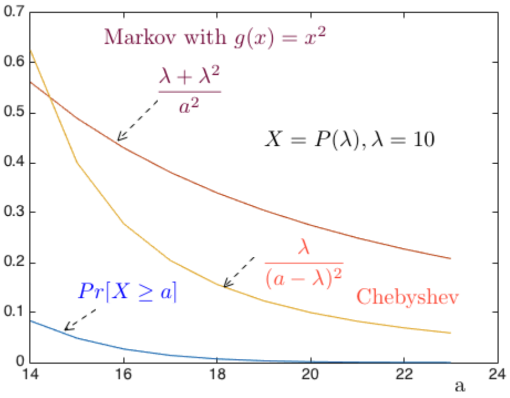
Hence, for $a > \lambda$, $Pr[X \geq a] \leq Pr[|X - \lambda| \geq a - \lambda] \leq \frac{\lambda}{(a-\lambda)^2}$.



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We look at a calculation of this next.

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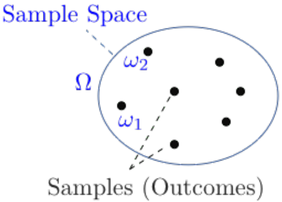
Probability: Midterm 2 Review.

- Framework:
  - Probability Space
  - Conditional Probability & Bayes' Rule
  - Independence
  - Mutual Independence

Review: Probability Space



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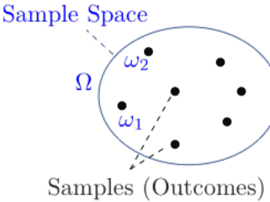

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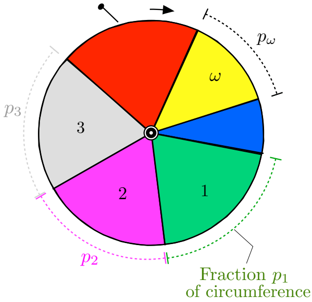

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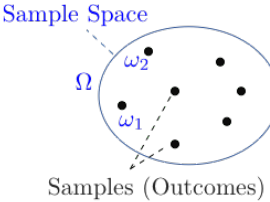

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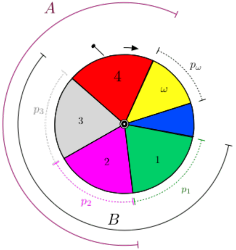

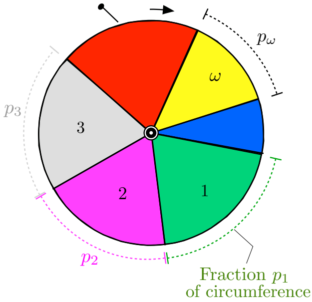

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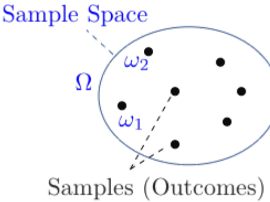

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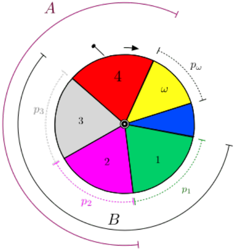

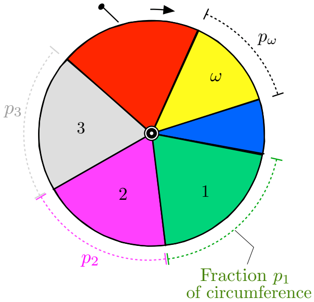

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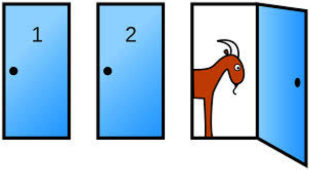

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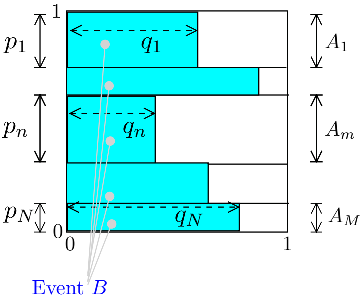

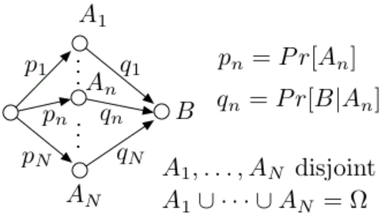

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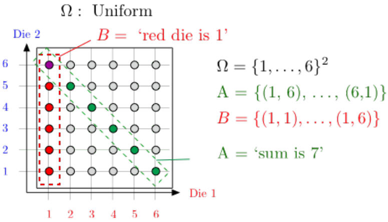

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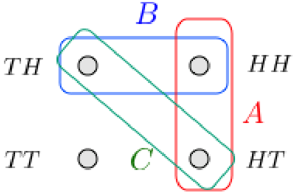
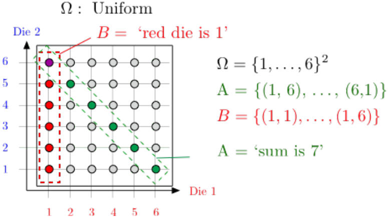


Pairwise, but not mutually



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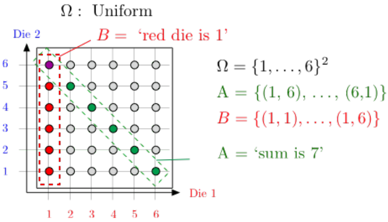

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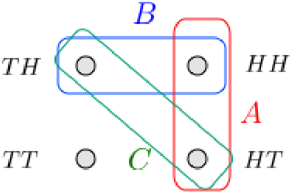

Pairwise, but not mutually

If $\{A_j, i \in J\}$ are mutually independent, then $[A_1 \cap \bar{A}_2] \Delta A_3$ and $A_4 \setminus A_5$ are independent.

Our intuitive meaning of "independent events" is mutual independence.

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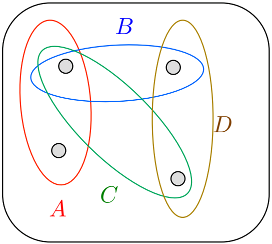

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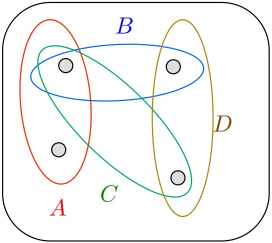

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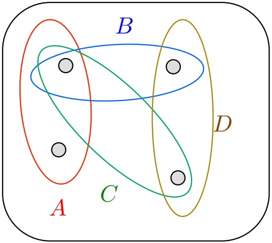

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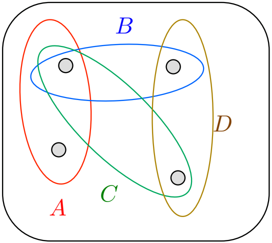

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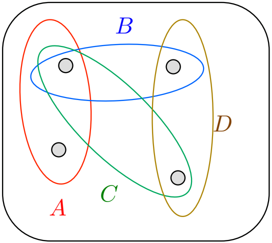

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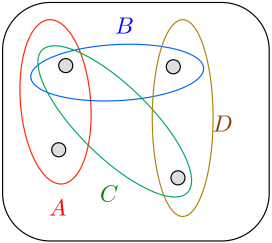

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No: We would need an outcome with probability $1/8$.

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Discrete Math:Review

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$x^{ed} - x$ is divisible by p and q $\implies$ theorem!

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Welsh-Berlekamp

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Counting

First Rule

Counting

First Rule
Second Rule

Counting

First Rule
Second Rule
Stars/Bars

Counting

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Second Rule
Stars/Bars
Common Scenarios: Sampling, Balls in Bins.

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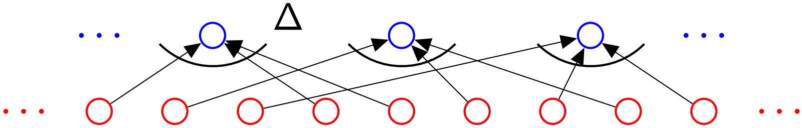
Example: visualize.

First rule: $n_1 \times n_2 \cdots \times n_3$. **Product Rule.**
Second rule: when order doesn't matter divide..when possible.



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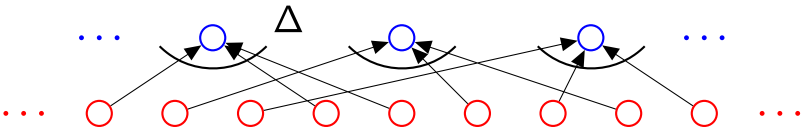

3 card Poker deals: 52

Example: visualize.

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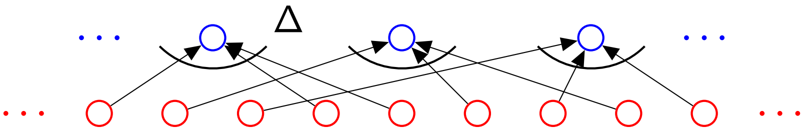

3 card Poker deals: 52×51

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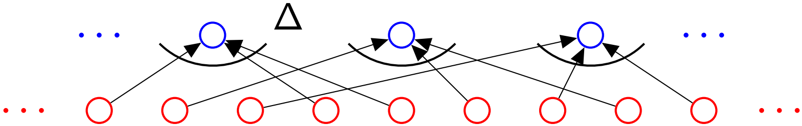

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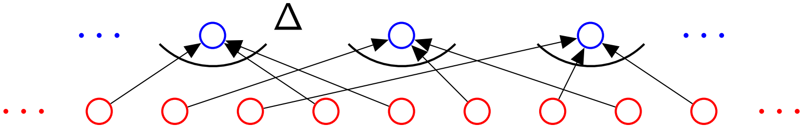

3 card Poker deals: $52 \times 51 \times 50 = \frac{52!}{49!}$.

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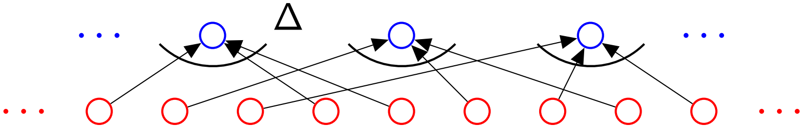

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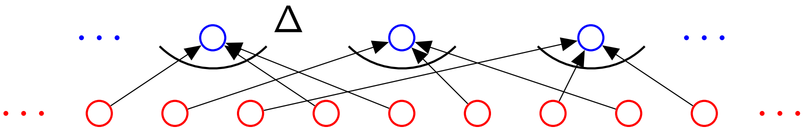

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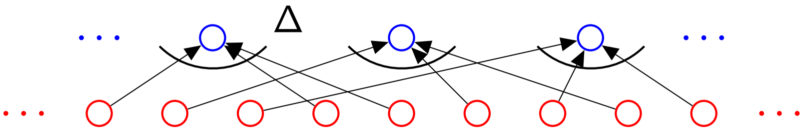

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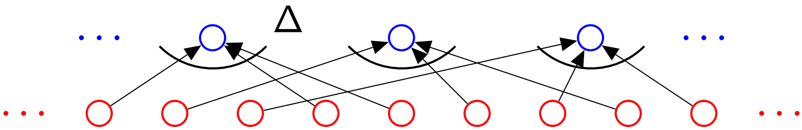

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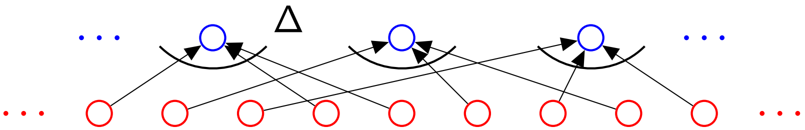

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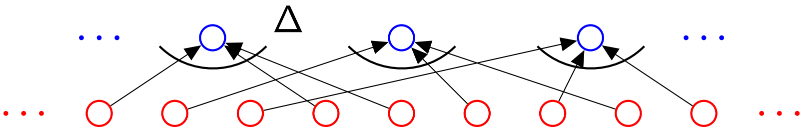

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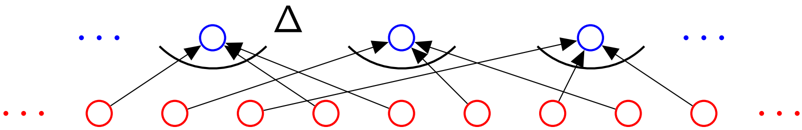

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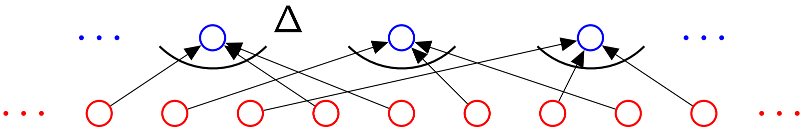

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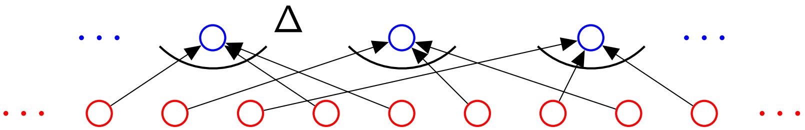

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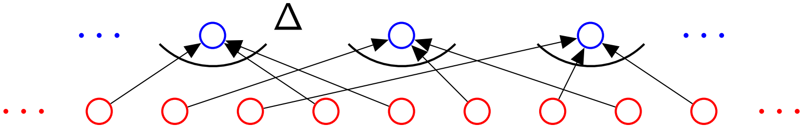

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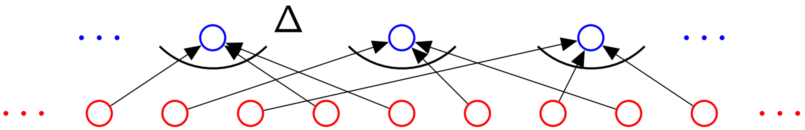

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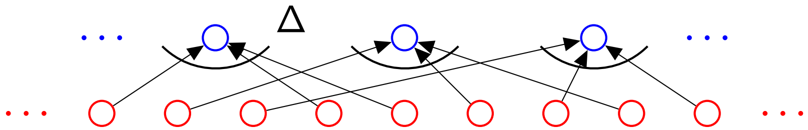

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Choose k out of n.

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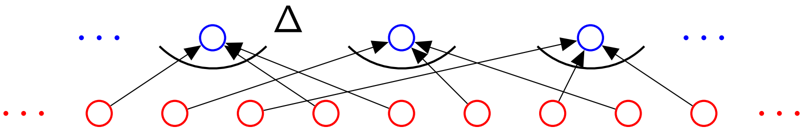

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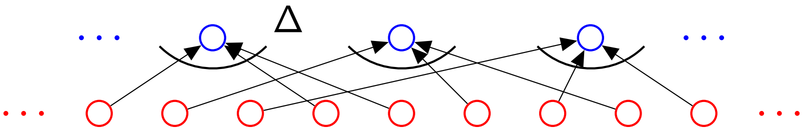

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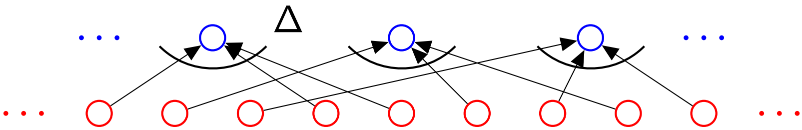

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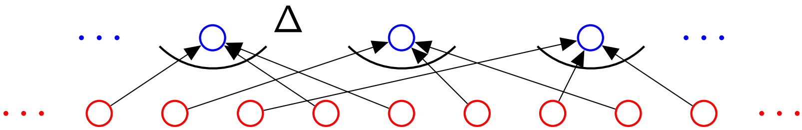

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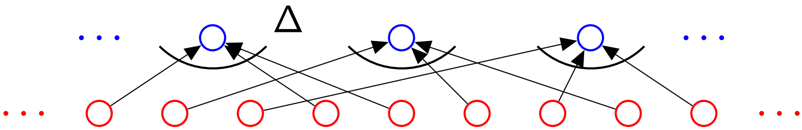

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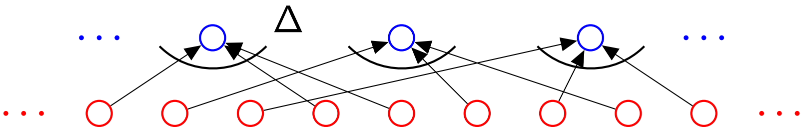

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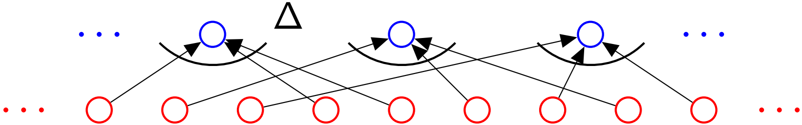

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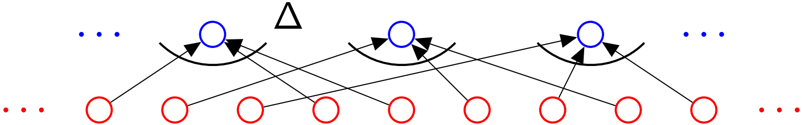

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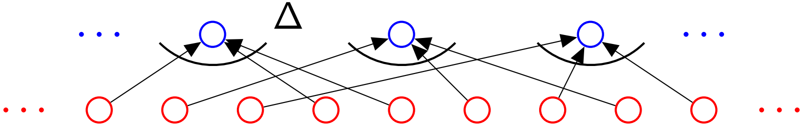

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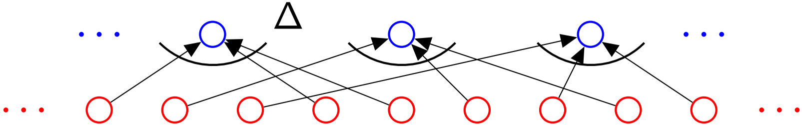

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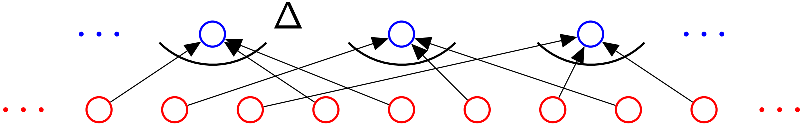

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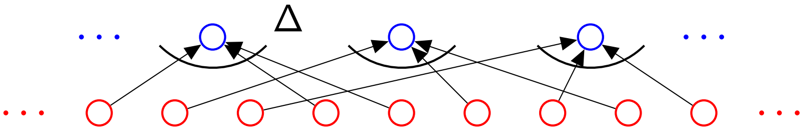

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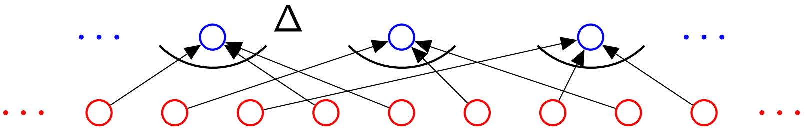

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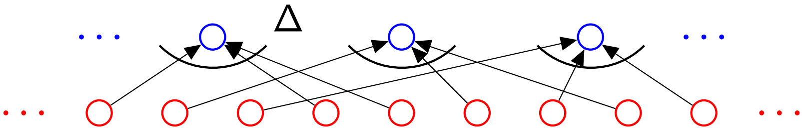

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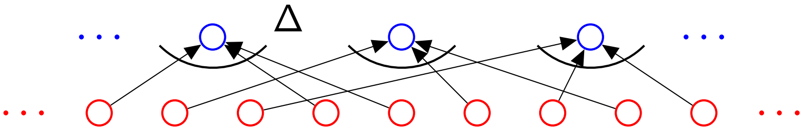

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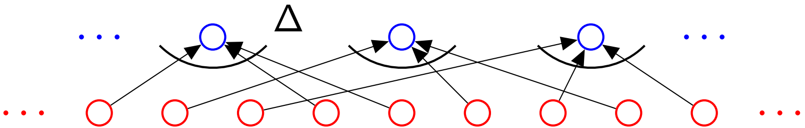

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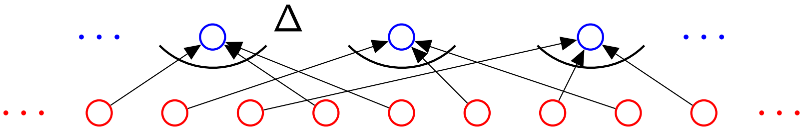

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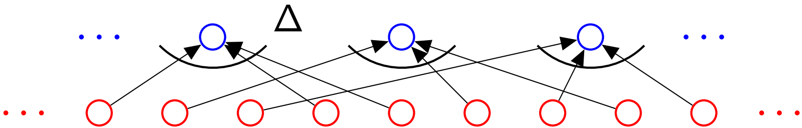

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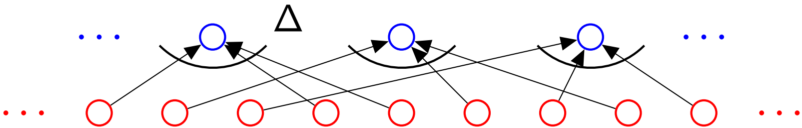

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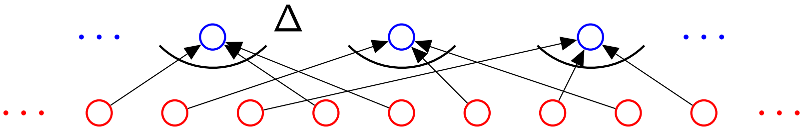

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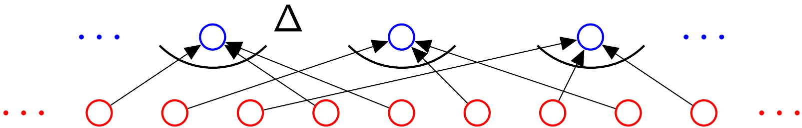

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Answer: $|S| + |T| - |S \cap T| = 10^9 + 10^9 - 10^8$.

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Proof: How many size k subsets of $n+1$? $\binom{n+1}{k}$.

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Isomporphism principle.

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$[0,1]$ is same cardinality as nonnegative reals!

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All countably infinite sets are the same cardinality as each other.

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Countably infinite (same cardinality as naturals)

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- Positive Rational numbers.
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Examples: Countable by enumeration

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Diagonalization: power set of Integers.

The set of all subsets of N.

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Another view: diagonalization.

Any program is a fixed length string.

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P_1	H	H	L	$\cdots$
P_2	L	L	H	$\cdots$
P_3	L	H	H	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

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Halt does not exist!

Undecidable problems.

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Find exit points and add statement: **Print** “Hello World.”

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Undecidability for Diophantine set of equations
$\implies$ no program can take any set of integer equations
  and always output correct answer.

Midterm format

Time: approximately 120 minutes.

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Many short answers.

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Many short answers.
Get at ideas that we study.

Midterm format

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Get at ideas that we study.
Know material well:

Midterm format

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Get at ideas that we study.

Know material well: fast,

Midterm format

Time: approximately 120 minutes.

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Get at ideas that we study.
Know material well: fast, correct.

Midterm format

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   Know material well: fast, correct.
   Know material medium:

Midterm format

Time: approximately 120 minutes.

Many short answers.
  Get at ideas that we study.
   Know material well: fast, correct.
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Time: approximately 120 minutes.

Many short answers.
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Wrapup.

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If you sent me email about Midterm conflicts

Wrapup.

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satishr@cs.berkeley.edu, admin@cs70.org

Wrapup.

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