CS70: Alex Psomas: Lecture 19.

1. Random Variables: Brief Review
2. Some details on distributions: Geometric. Poisson.
3. Joint distributions.
4. Linearity of Expectation.

Random Variables: Definitions

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where $\mathscr{A}$ is the *range* of X. That is, $\mathscr{A} = \{X(\omega), \omega \in \Omega\}$.

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$$Pr[X = 2] = Pr[X^{-1}(2)] = Pr[\{HHT, HTH, THH\}]$$

$$= Pr[\{HHT\}] + Pr[\{HTH\}] + Pr[\{THH\}] = \frac{3}{8}$$

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Also,

$$E[X] = \sum_{a \in \mathbb{R}} a \times Pr[X = a] = 3 \times \frac{1}{8} + 2 \times \frac{3}{8} + 1 \times \frac{3}{8} + 0 \times \frac{1}{8}.$$

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the Law of Large Numbers. (See later.)

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Note that $\Omega = \{\omega_n, n = 1, 2, \ldots\}$. (Notice: no distribution yet!)

Let X be the number of flips until the first H. Then, $X(\omega_n) = n$.

Also,

$$Pr[X = n] =$$

Geometric Distribution

Let's flip a coin with $Pr[H] = p$ until we get H.

For instance:

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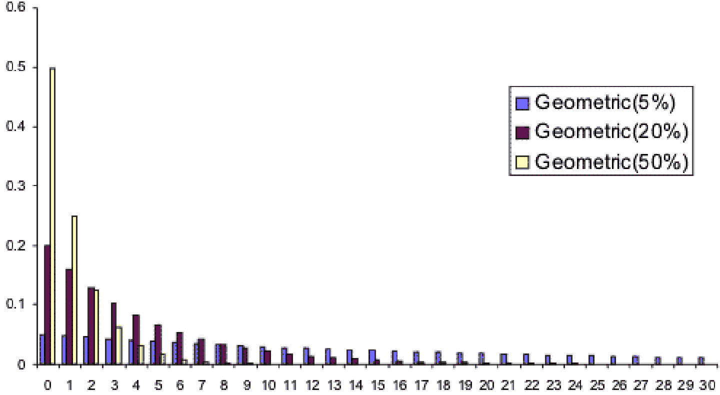

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Hence,

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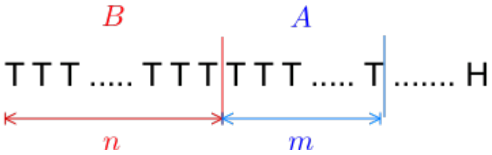

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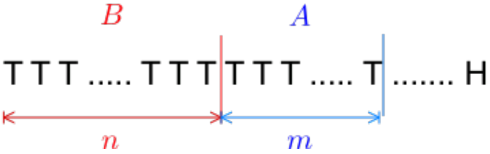

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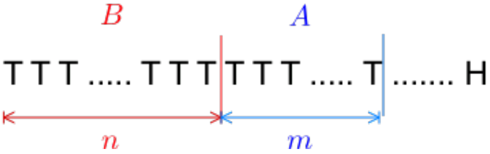

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The coin is memoryless, therefore, so is X.

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Theorem: For a r.v. X that takes the values $\{0, 1, 2, \ldots\}$, one has

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\end{aligned}$$

Expected Value of Integer RV

Theorem: For a r.v. X that takes values in $\{0, 1, 2, \ldots\}$, one has

$$E[X] = \sum_{i=1}^{\infty} Pr[X \geq i].$$

Proof: One has

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Poisson Distribution: Definition and Mean

Definition Poisson Distribution with parameter $\lambda > 0$

$$X = P(\lambda) \Leftrightarrow Pr[X = m] = \frac{\lambda^m}{m!} e^{-\lambda}, m \geq 0.$$

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Simeon Poisson

The Poisson distribution is named after:

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Siméon Denis Poisson (1781–1840)

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Thus, we will write $X = 1_A$.

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Two random variables, X and Y, in prob space: $(\Omega, P(\cdot))$.

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Important for inference.

Two random variables, same outcome space.

Experiment: pick a random person.

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X =	0	1	2	3	5	40	All
Pr	0.3	0.05	0.05	0.05	0.05	0.1	0.4

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X =	0	1	2	3	5	40	All
Pr	0.3	0.05	0.05	0.05	0.05	0.1	0.4

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Joint distribution: Example.

The **joint distribution** of X and Y is:

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5	0	0	0	0.05	0.05	0	0	=0.1
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But now we have more information!

For example, if I tell you someone watched 5 episodes of Westworld, they definitely didn't watch all the episodes of GoT.

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- $X \cos(2\pi Y + Z)$.

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Note: If we had defined $Y = a_1X_1 + \cdots + a_nX_n$ and had tried to compute $E[Y] = \sum_y yPr[Y = y]$, we would have been in trouble!

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Roll a die n times.

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Note: Computing $\sum_x xPr[X = x]$ directly is not easy!

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Note that linearity holds even though the X_m are not independent (whatever that means).

Using Linearity - 2: Fixed point.

Hand out assignments at random to n students.

$X =$ number of students that get their own assignment back.

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Flip n coins with heads probability p.

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Flip n coins with heads probability p. X - number of heads

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$\square$

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Y = \begin{cases} 4, & \text{w.p. } \frac{2}{6} \\ 1, & \text{w.p. } \frac{2}{6} \\ 0, & \text{w.p. } \frac{1}{6} \\ 9, & \text{w.p. } \frac{1}{6}. \end{cases}
$$

An Example

Let X be uniform in $\{-2, -1, 0, 1, 2, 3\}$.

Let also $g(X) = X^2$. Then (method 2)

$$\begin{aligned} E[g(X)] &= \sum_{x=-2}^{3} x^2 \frac{1}{6} \\ &= \{4+1+0+1+4+9\}\frac{1}{6} = \frac{19}{6}. \end{aligned}$$

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Thus,

$$E[Y] = 4\frac{2}{6} + 1\frac{2}{6} + 0\frac{1}{6} + 9\frac{1}{6} = \frac{19}{6}.$$

Summary

Random Variables

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Random Variables

- A random variable X is a function $X : \Omega \to \Re$.
- $Pr[X = a] := Pr[X^{-1}(a)] = Pr[\{\omega \mid X(\omega) = a\}]$.
- $Pr[X \in A] := Pr[X^{-1}(A)]$.
- The distribution of X is the list of possible values and their probability: $\{(a, Pr[X = a]), a \in \mathscr{A}\}$.
- Joint distributions.
- $g(X, Y, Z)$ assigns the value .... .
- $E[X] := \sum_a a Pr[X = a]$.
- Expectation is Linear.