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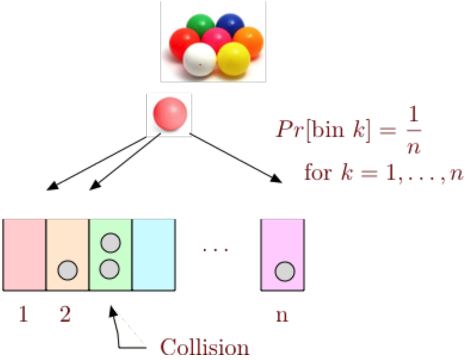

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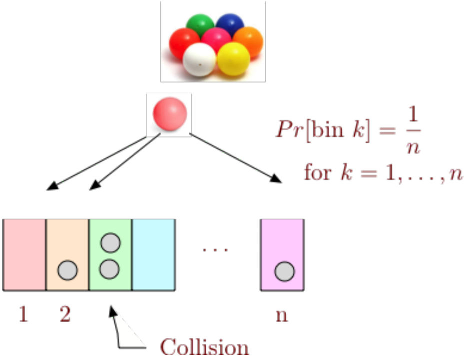

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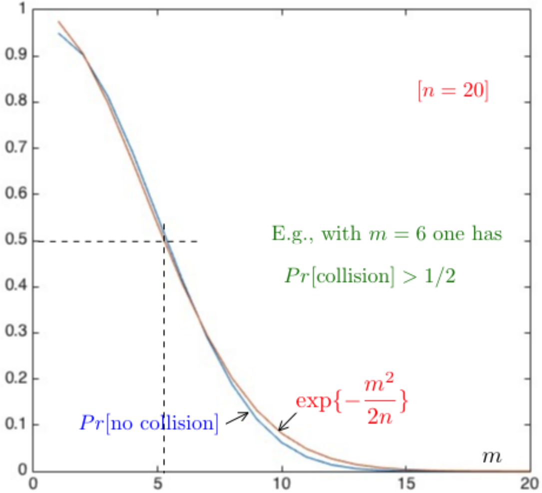

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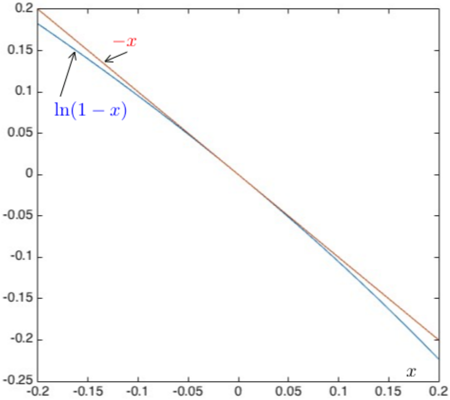
(†) $1 + 2 + \cdots + m - 1 = (m-1)m/2$.

Approximation



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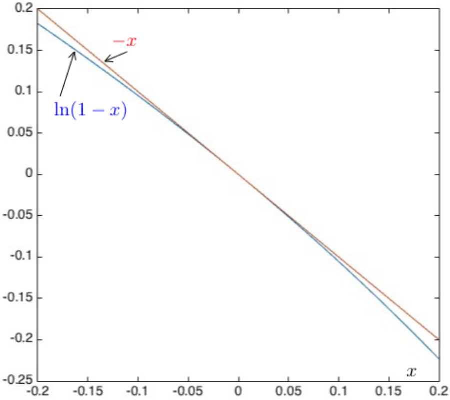

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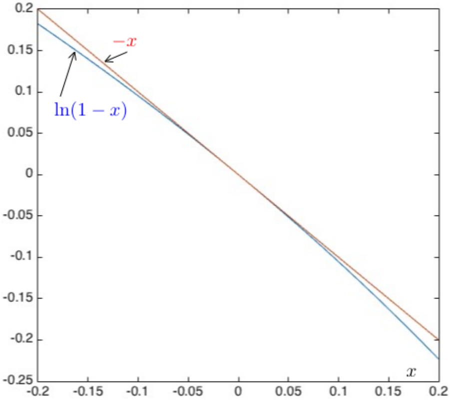

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Hence, $-x \approx \ln(1-x)$ for $|x| \ll 1$.

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Note: $\log_2(x) = \log_2(e)\ln(x) \approx 1.44\ln(x)$.

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Pr[A_m] &\approx \exp\{-\frac{m}{n}\}.
\end{aligned}
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For $p_m = \frac{1}{2}$, we need around $n \ln 2 \approx 0.69n$ boxes.

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Plug in and get

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1. Random Variables.
2. Expectation
3. Distributions.

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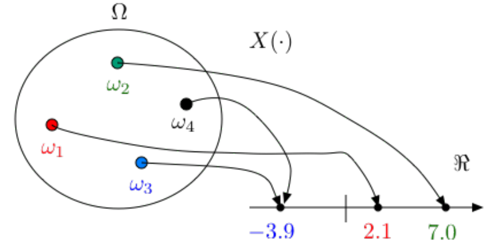

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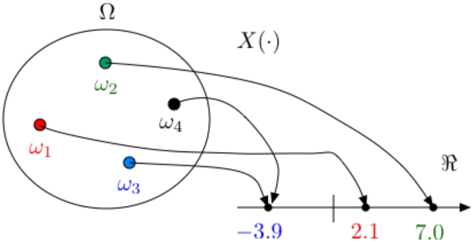

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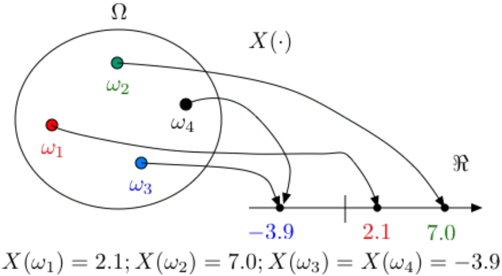

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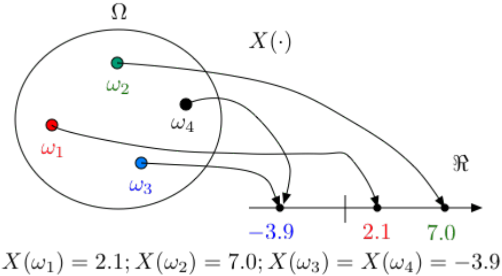

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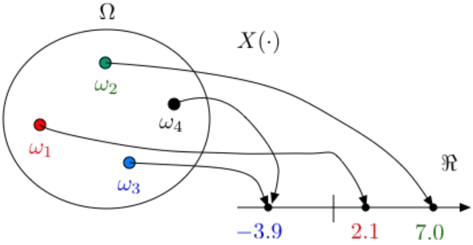

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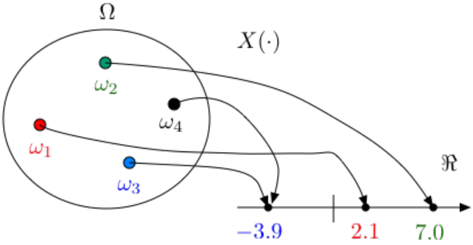

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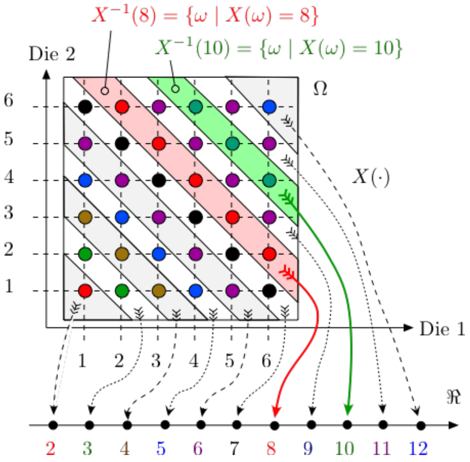

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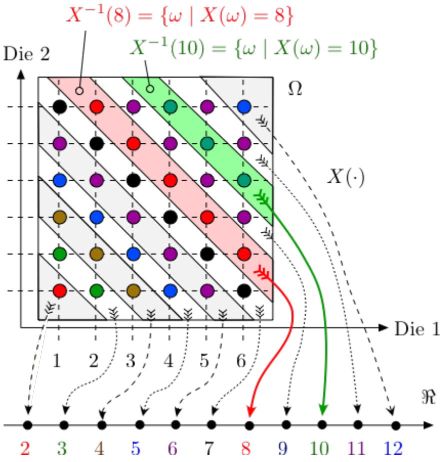

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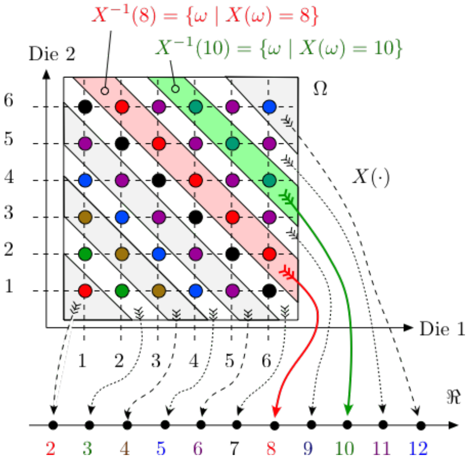

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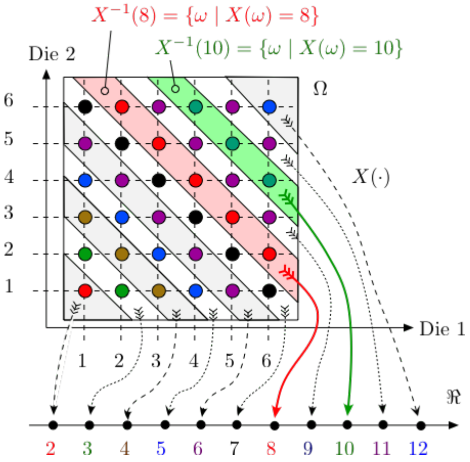

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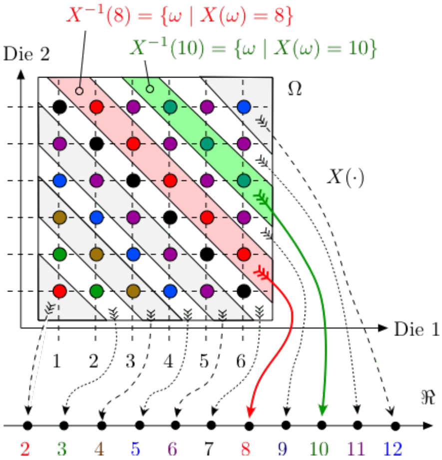

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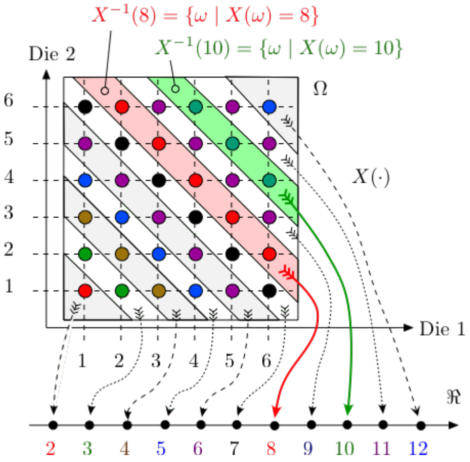

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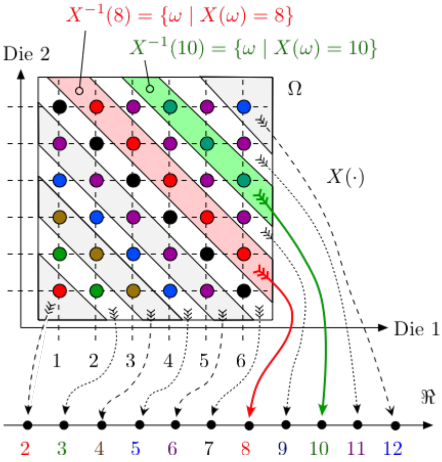

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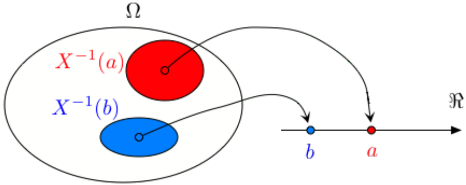

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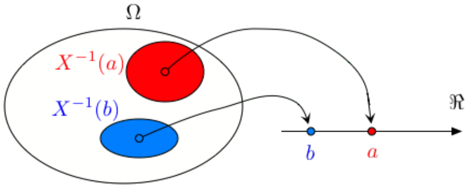

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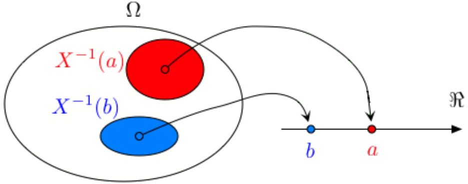

$$Pr[X = a] := Pr[X^{-1}(a)] \text{ where } X^{-1}(a) := \{\omega \mid X(\omega) = a\}.$$

Handing back assignments

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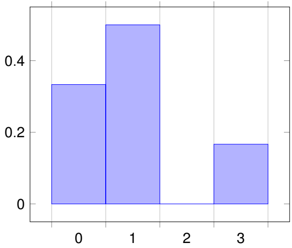

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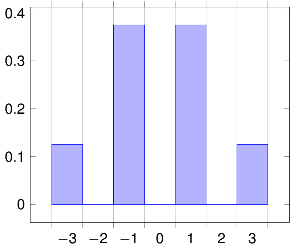

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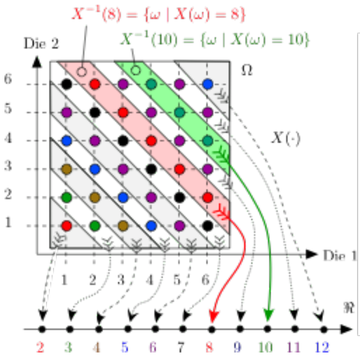

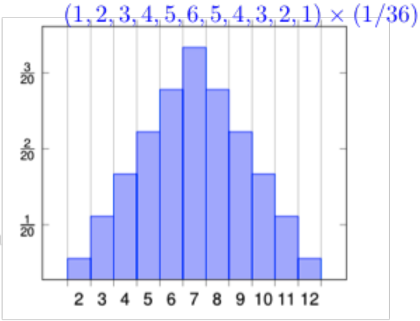

Expectation.

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Summary of distribution?

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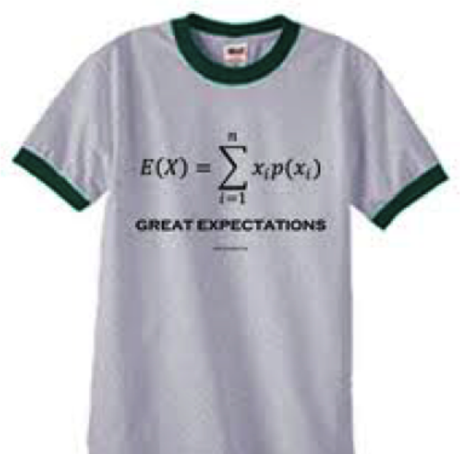

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The subjectivist(bayesian) interpretation of $E[X]$ is less obvious.

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Distributive property of multiplication over addition.

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Let’s cover some.

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Probability of "$X = i$" is sum of $Pr[\omega]$, $\omega \in$ "$X = i$".

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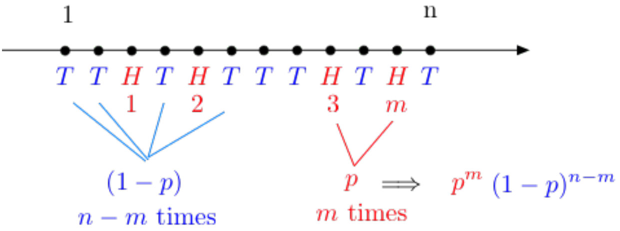

$\binom{n}{m}$ outcomes with m Hs and $n-m$ Ts

$$\Longrightarrow \boxed{Pr[X=m] = \binom{n}{m} p^m (1-p)^{n-m}}$$

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Also distribution in polling, experiments, etc.

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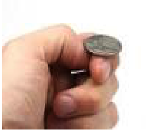

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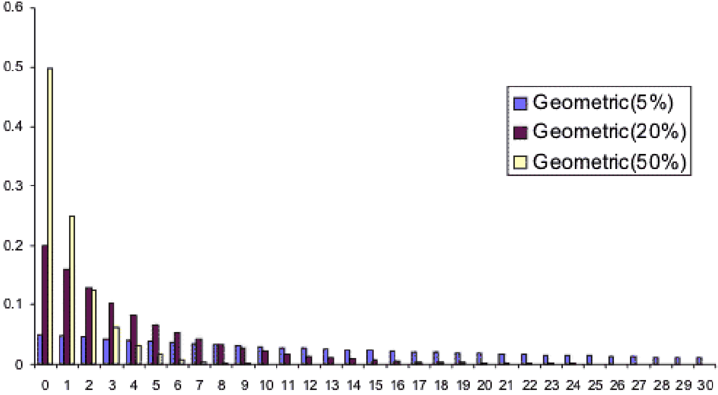

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$$E[X] = \frac{1}{p}.$$

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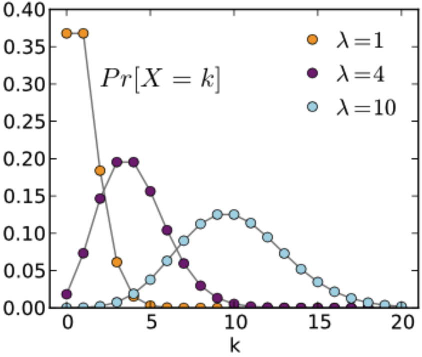

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$\square$

Simeon Poisson

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Equal Time: B. Geometric

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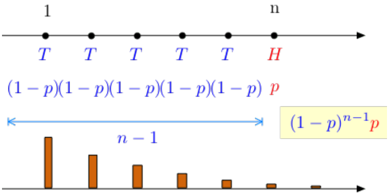

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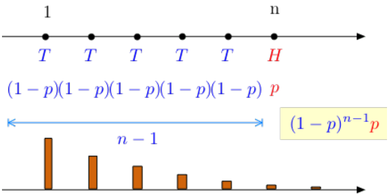

I could not find a picture of D. Binomial, sorry.

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