Lecture 16: Continuing Probability.

Wrap up: Probability Formalism.

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Events, Conditional Probability, Independence, Bayes' Rule

Probability Space: Formalism

Simplest physical model of a uniform probability space:

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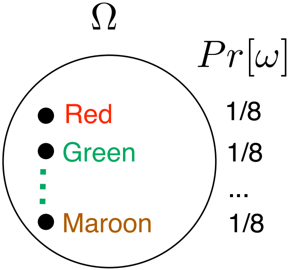
Physical experiment



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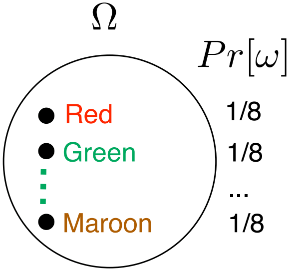

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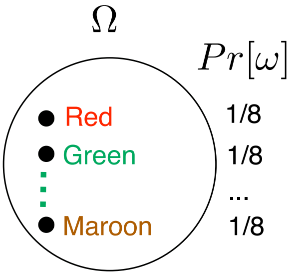

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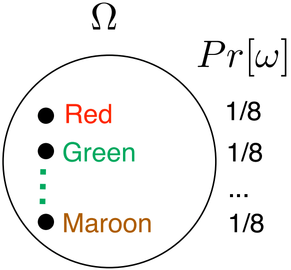

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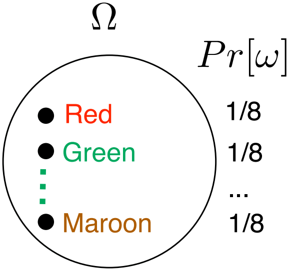

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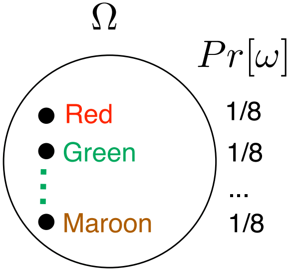

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$$Pr[\text{blue}] = \frac{1}{8}.$$

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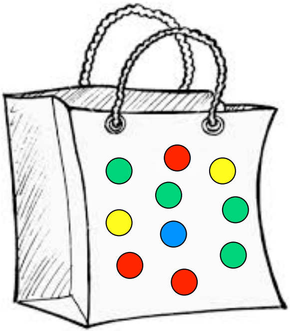

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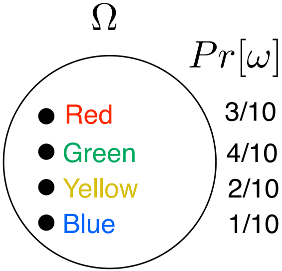

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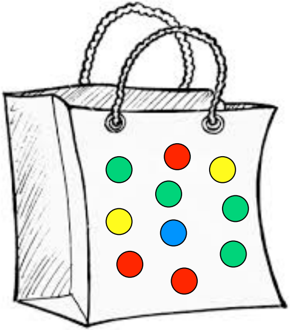

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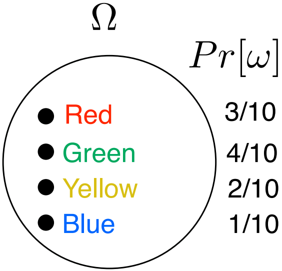

Probability model

$\Omega = \{$Red, Green, Yellow, Blue$\}$

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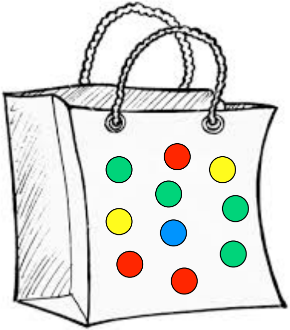

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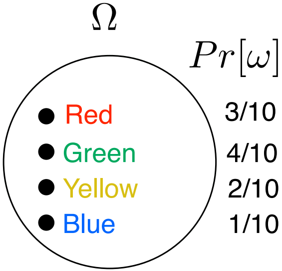

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$Pr[\text{Red}] =$

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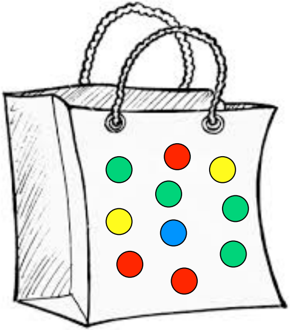

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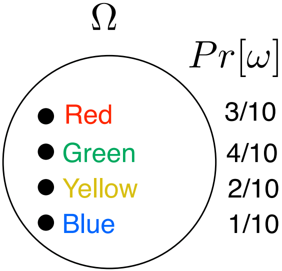

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$$\Omega = \{\text{Red, Green, Yellow, Blue}\}$$

$$Pr[\text{Red}] = \frac{3}{10},$$

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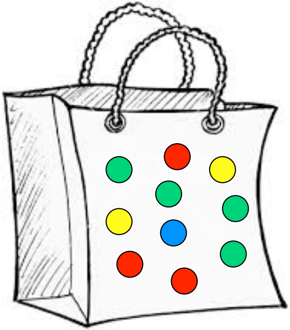

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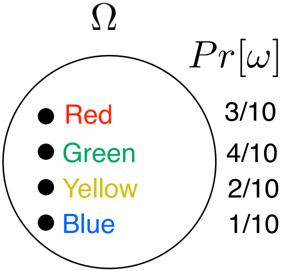

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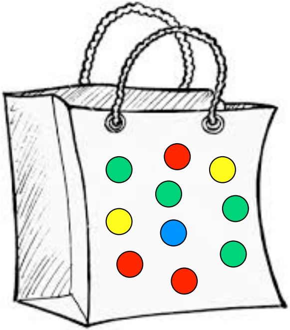
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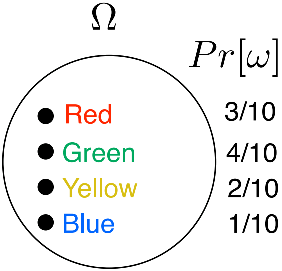

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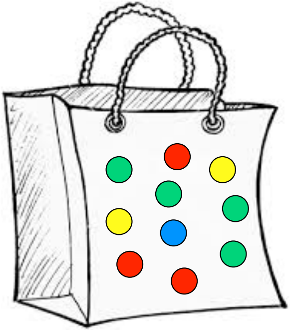
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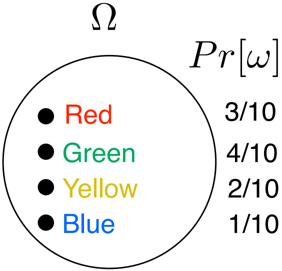

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Note: Probabilities are restricted to rational numbers: $\frac{N_k}{N}$.

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Physical model of a general non-uniform probability space:

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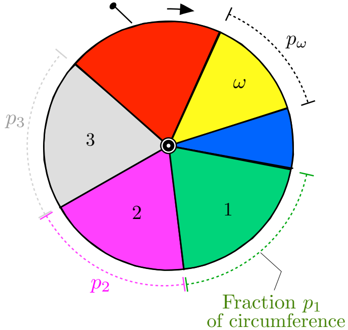

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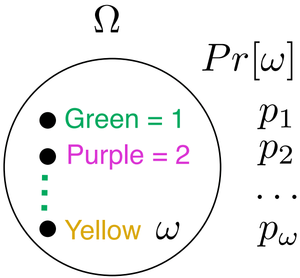

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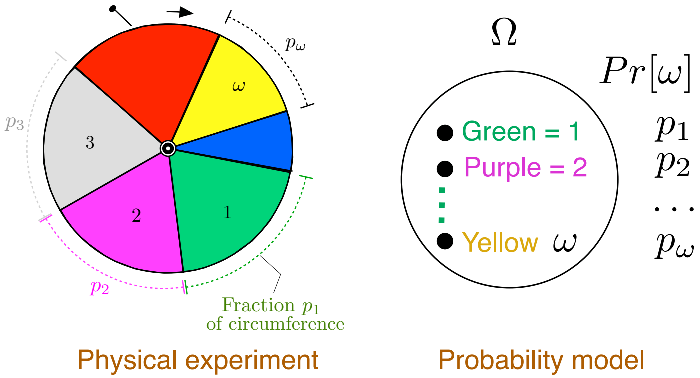

The roulette wheel stops in sector ω with probability p_ω.

Probability Space: Formalism

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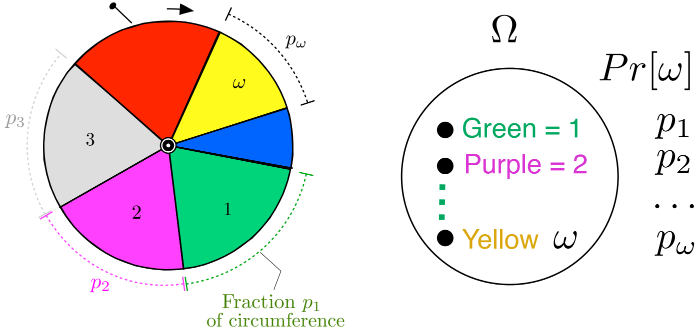

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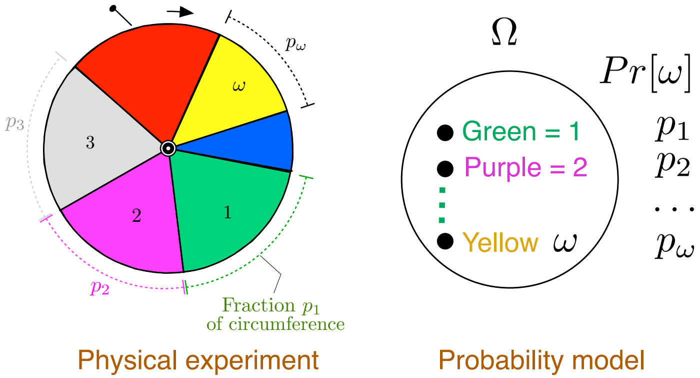

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Lecture 15: Summary

Modeling Uncertainty: Probability Space

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CS70: On to Calculation.

Events, Conditional Probability, Independence, Bayes' Rule

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Events, Conditional Probability, Independence, Bayes' Rule

1. Probability Basics Review
2. Events
3. Conditional Probability
4. Independence of Events
5. Bayes' Rule

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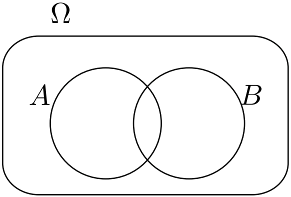
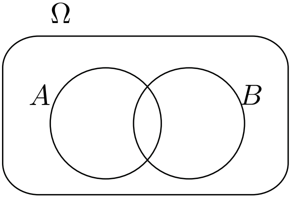


Figure : Two events

Set notation review



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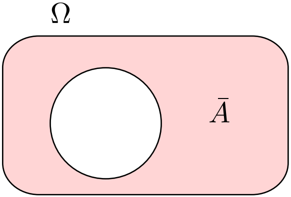

Figure : Complement (not)

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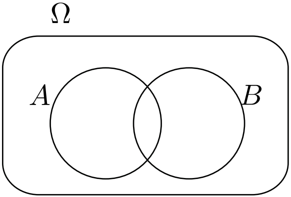

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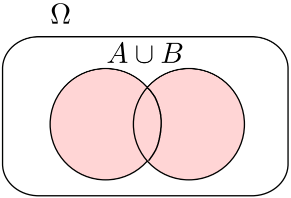
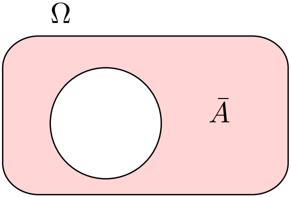


Figure : Union (or)



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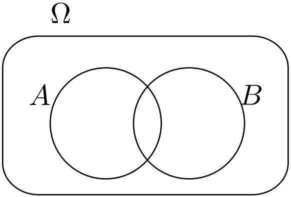

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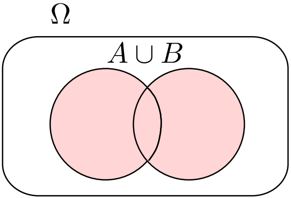

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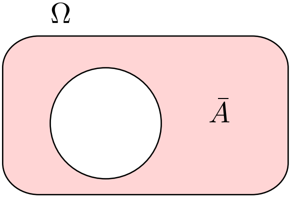

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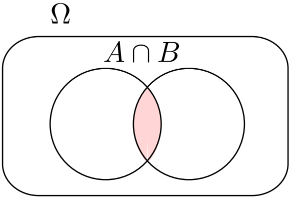

Figure : Intersection (and)

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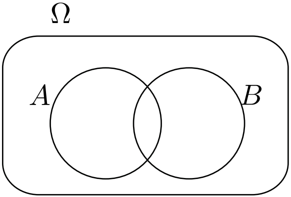

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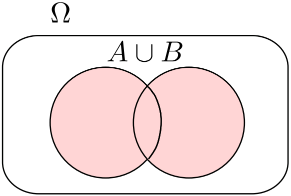

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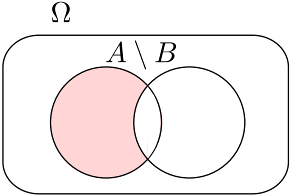
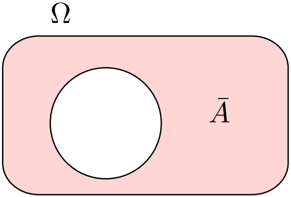


Figure : Difference (*A*, not *B*)



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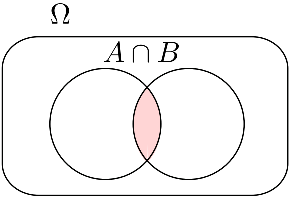

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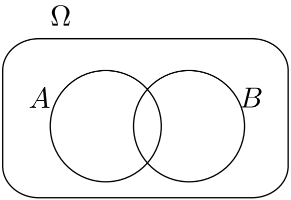

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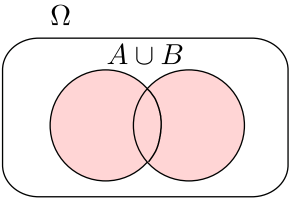

Figure : Union (or)

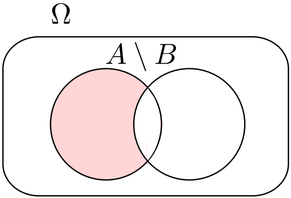

Figure : Difference (*A*, not *B*)

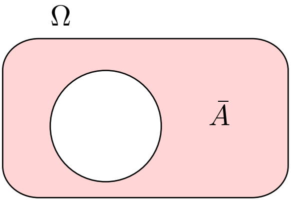

Figure : Complement (not)

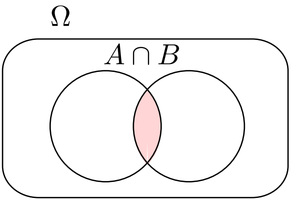

Figure : Intersection (and)

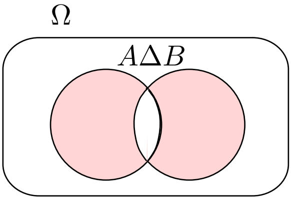

Figure : Symmetric difference (only one)

Probability of exactly one ‘heads’ in two coin flips?

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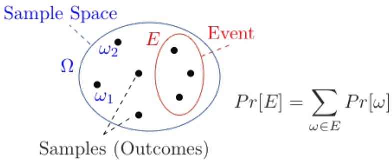

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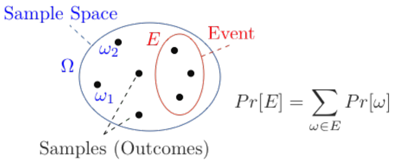

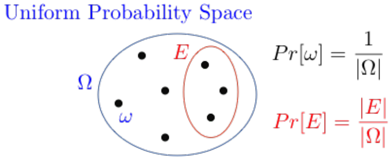

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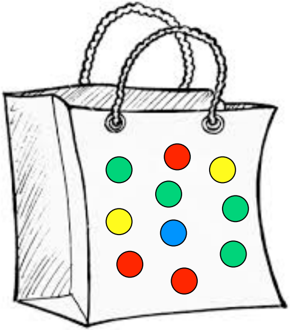
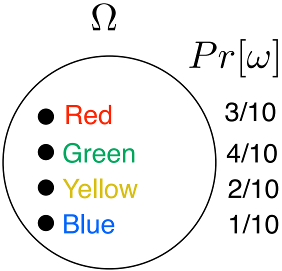


Physical experiment



Probability model

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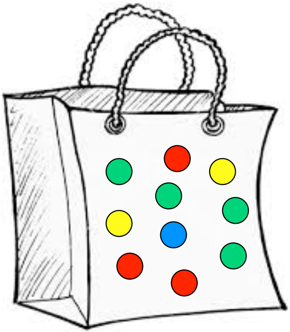

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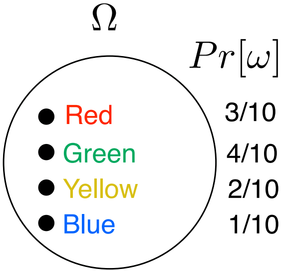

Probability model

$\Omega = \{$Red, Green, Yellow, Blue$\}$

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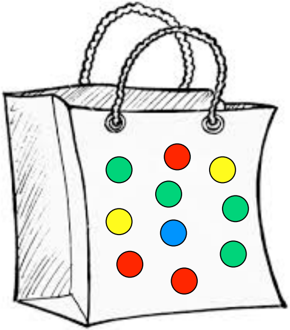

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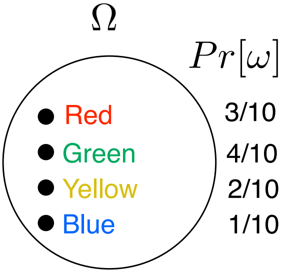

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$Pr[\text{Red}] =$

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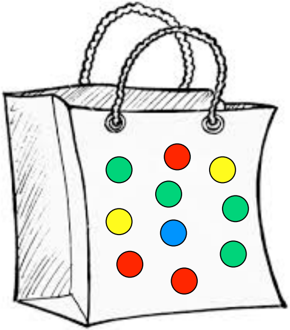

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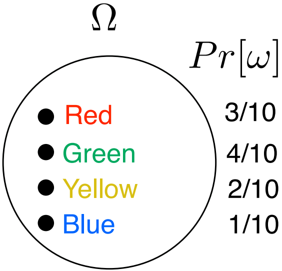

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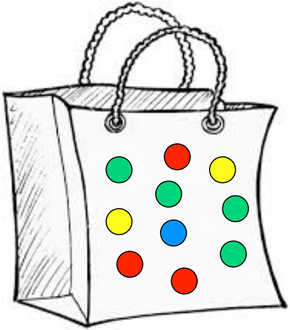

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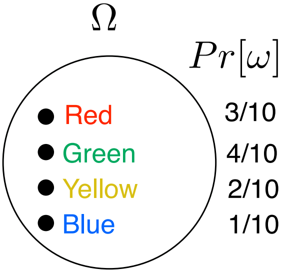

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$$\Omega = \{\text{Red, Green, Yellow, Blue}\}$$

$$Pr[\text{Red}] = \frac{3}{10}, Pr[\text{Green}] =$$

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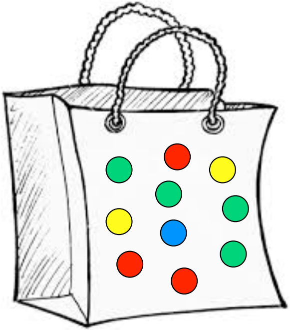

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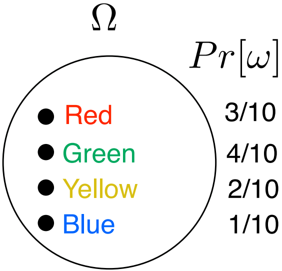

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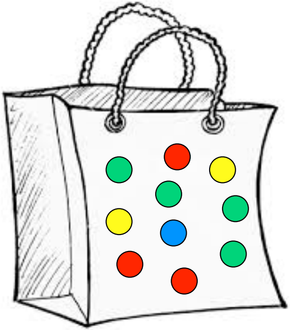

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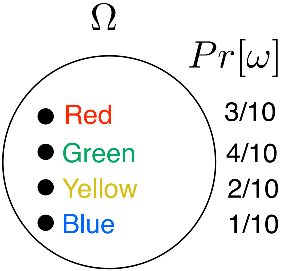

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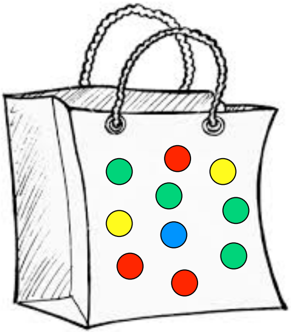

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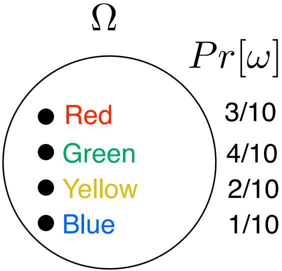

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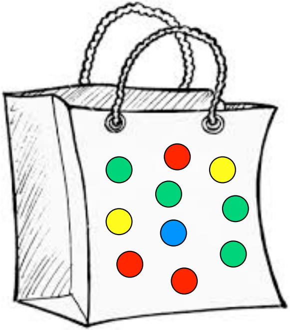

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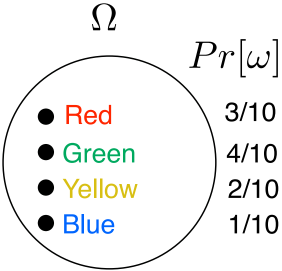

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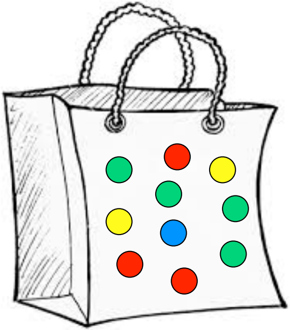

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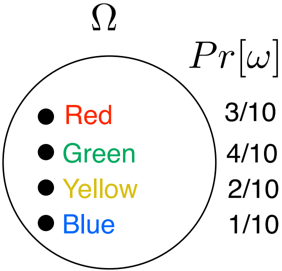

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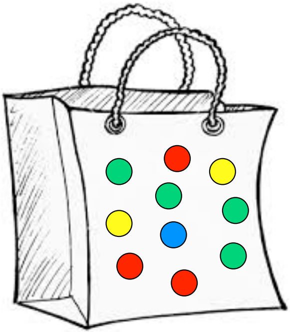

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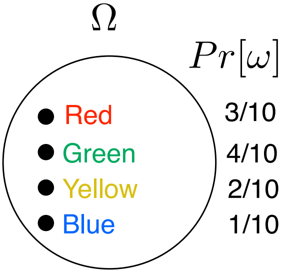

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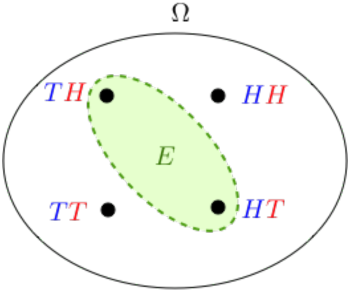

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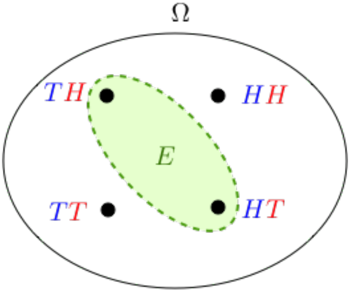

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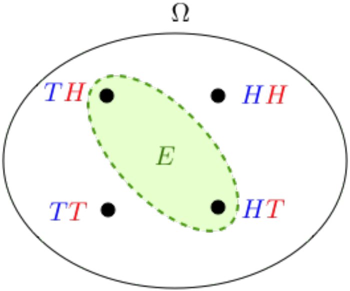

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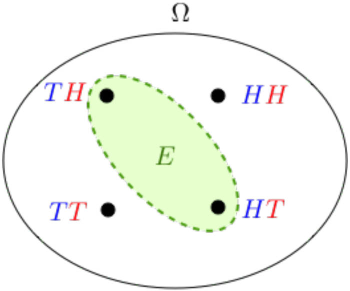

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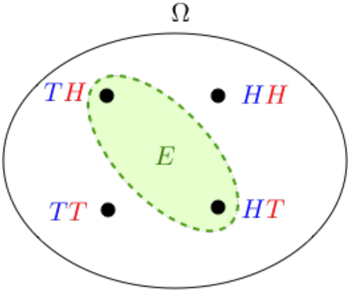

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  Why? There are many sequences of 20 tosses with ten Hs;

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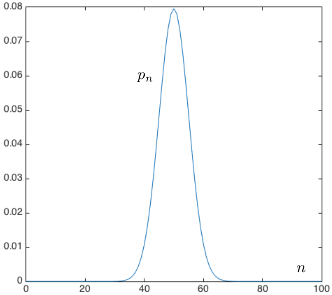

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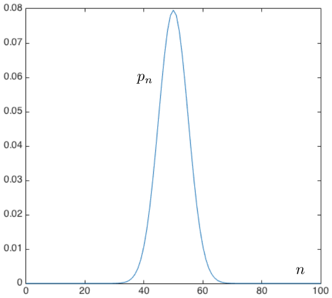

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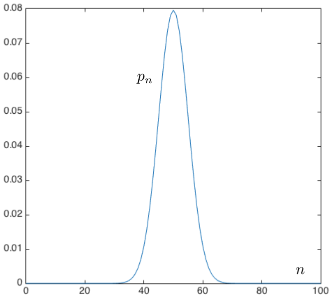

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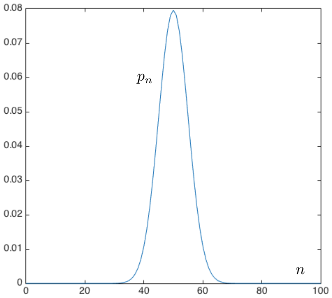

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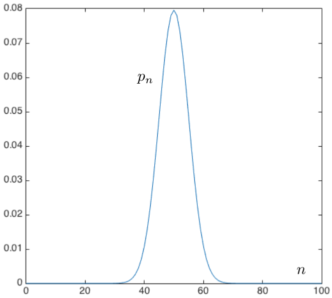

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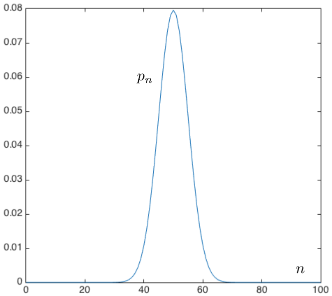

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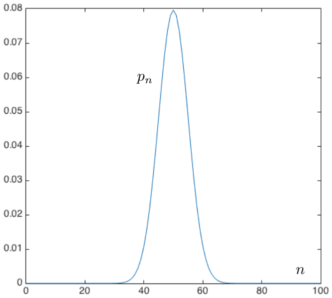

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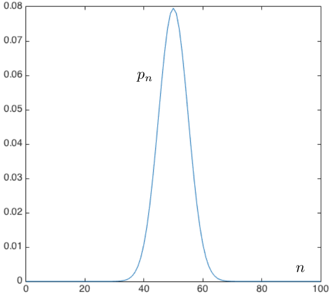

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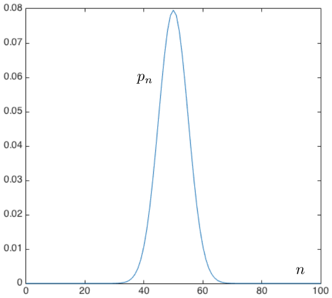

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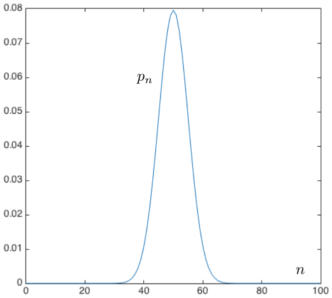

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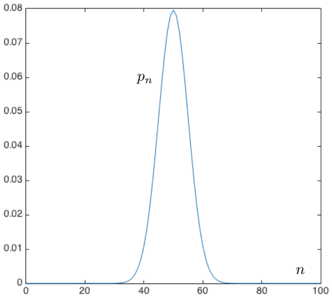

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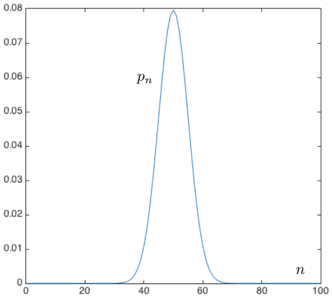

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- Bell-shape: Central Limit Theorem.

Roll a red and a blue die.

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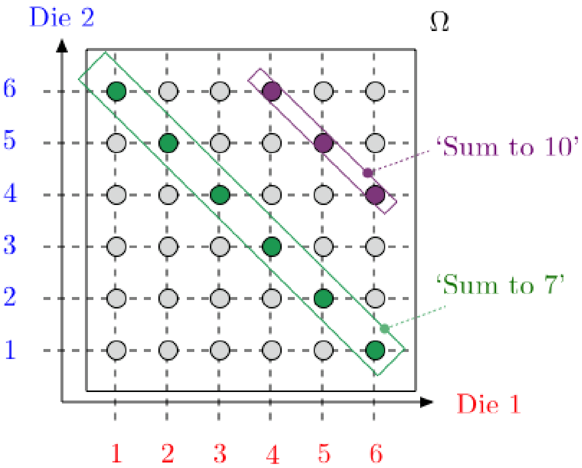

$$Pr[\text{Sum to } 7] = \frac{6}{36} \qquad Pr[\text{Sum to } 10] = \frac{3}{36}$$

Exactly 50 heads in 100 coin tosses.

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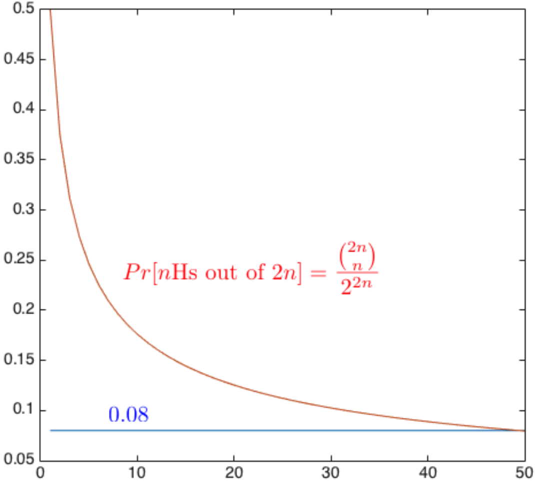

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Proofs for (a) and (c)?

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(law of total probability)

Proof:

(b) is obvious.

Proofs for (a) and (c)? Next...

Inclusion/Exclusion

$$Pr[A \cup B] = Pr[A] + Pr[B] - Pr[A \cap B]$$

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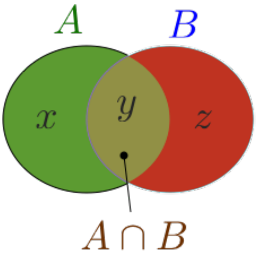

$$Pr[A] = x + y$$
$$Pr[B] = y + z$$
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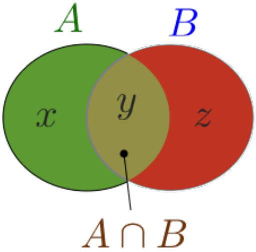

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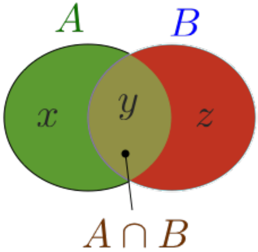
Another view.

Inclusion/Exclusion

$$Pr[A \cup B] = Pr[A] + Pr[B] - Pr[A \cap B]$$

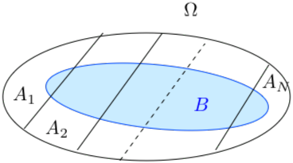


$$\begin{aligned} Pr[A] &= x + y \\ Pr[B] &= y + z \\ Pr[A \cap B] &= y \\ Pr[A \cup B] &= x + y + z \end{aligned}$$

Another view. Any $\omega \in A \cup B$ is in $A \cap \overline{B}$, $A \cup B$, or $\overline{A} \cap B$. So, add it up.

Total probability

Assume that Ω is the union of the disjoint sets $A_1, \ldots, A_N$.



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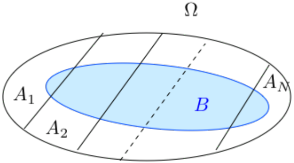

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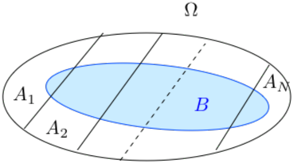

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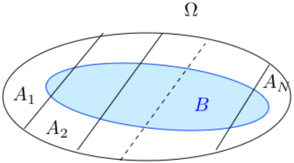

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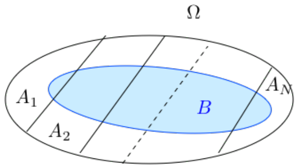

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Adding up probability of them, get $Pr[\omega]$ in sum.

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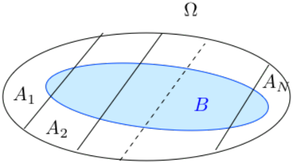

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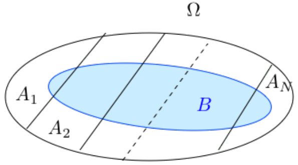

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Add it up.

Roll a Red and a Blue Die.

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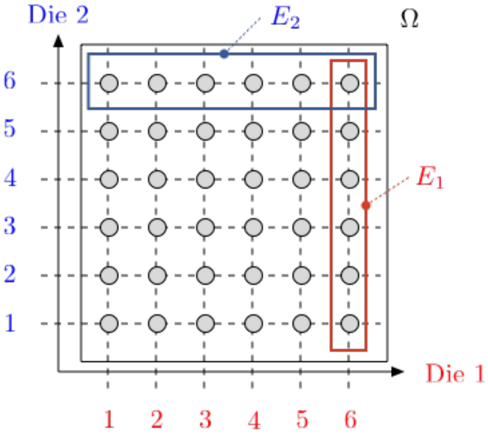

$$|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|$$

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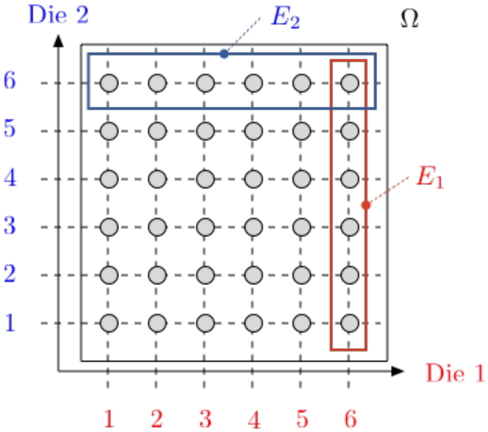

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$E_1 =$ ‘Red die shows 6’;

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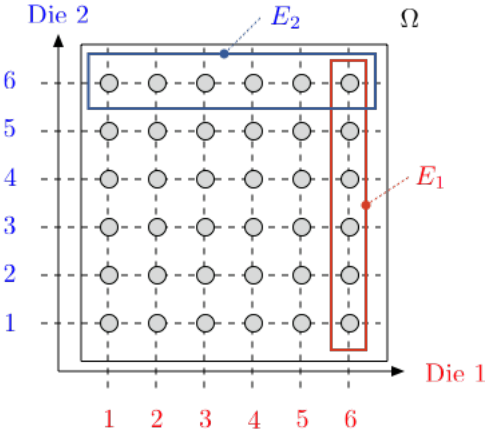

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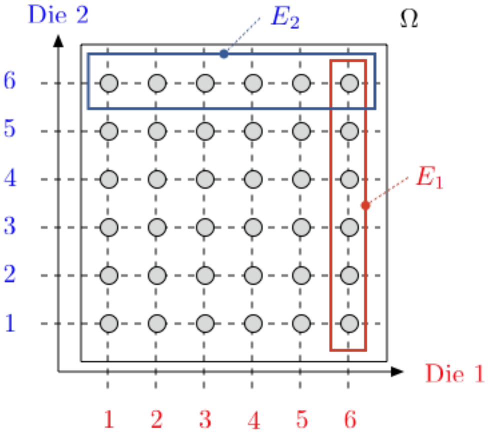

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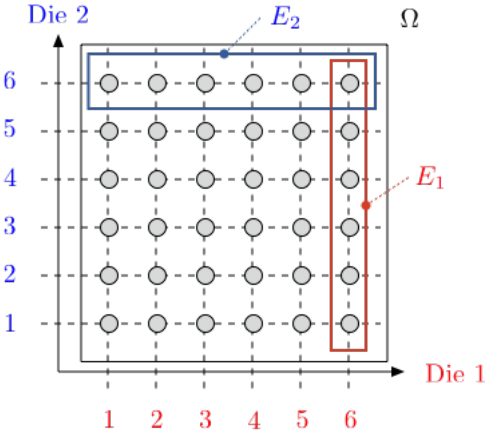

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$$Pr[E_1] = \frac{6}{36},$$

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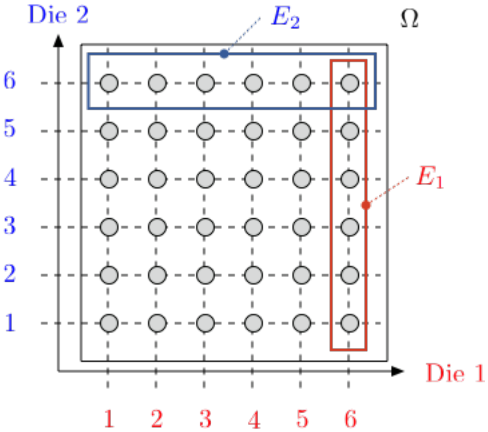

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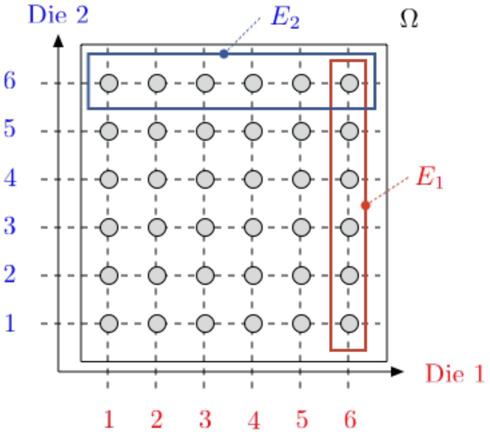

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E_1 = 'Red die shows 6'; E_2 = 'Blue die shows 6'

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$$Pr[E_1] = \frac{6}{36}, Pr[E_2] = \frac{6}{36}, Pr[E_1 \cup E_2] = \frac{11}{36}.$$

Conditional probability: example.

Two coin flips.

Conditional probability: example.

Two coin flips. First flip is heads.

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Two coin flips. First flip is heads. Probability of two heads?

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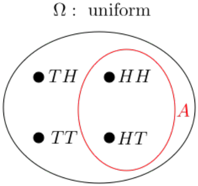

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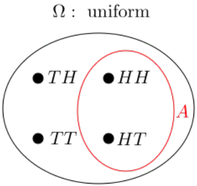

New sample space: A;

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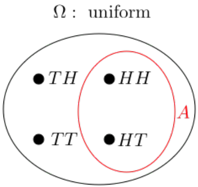

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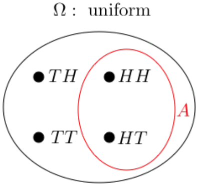

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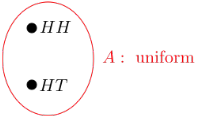

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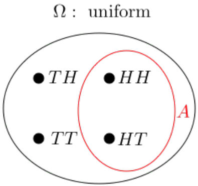

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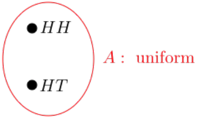

Event $B =$ two heads.

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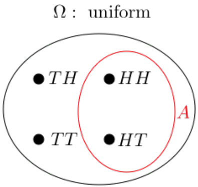

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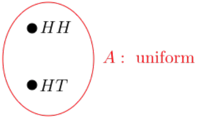

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The probability of two heads if the first flip is heads.

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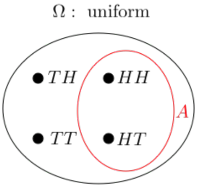

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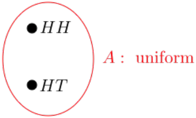

Event $B =$ two heads.

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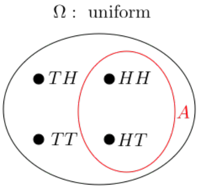

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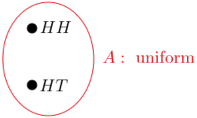

Event $B =$ two heads.

The probability of two heads if the first flip is heads.
The probability of B given A is $1/2$.

A similar example.

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Two coin flips. At least one of the flips is heads.

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$\rightarrow$ Probability of two heads?

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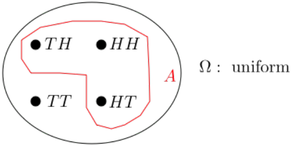

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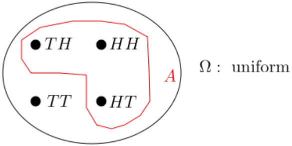

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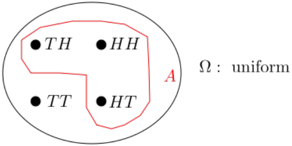

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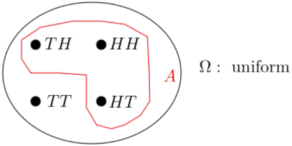

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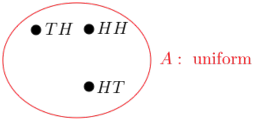

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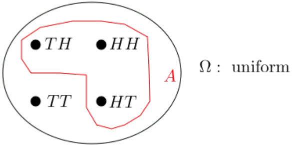

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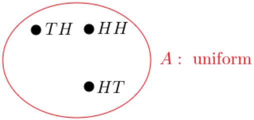

Event $B =$ two heads.

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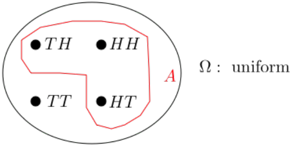

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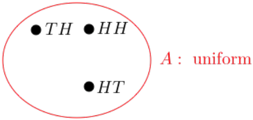

Event $B =$ two heads.

The probability of two heads if at least one flip is heads.

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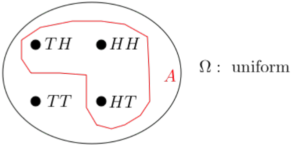

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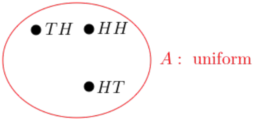

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The probability of B given A

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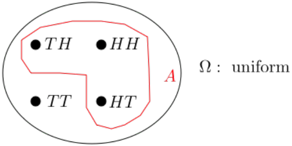
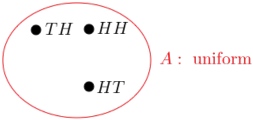
$\Omega = \{HH, HT, TH, TT\}$; uniform.
Event $A =$ at least one flip is heads. $A = \{HH, HT, TH\}$.



New sample space: A; uniform still.



Event $B =$ two heads.

The probability of two heads if at least one flip is heads.
The probability of B given A is $1/3$.

Conditional Probability: A non-uniform example

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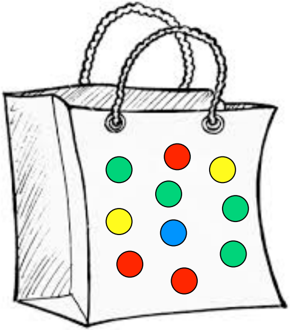
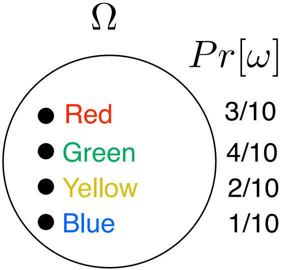
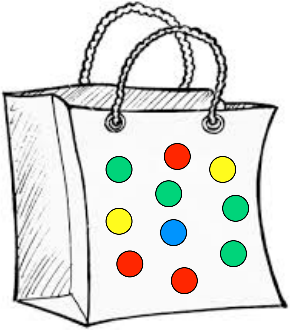


Physical experiment



Probability model

Conditional Probability: A non-uniform example



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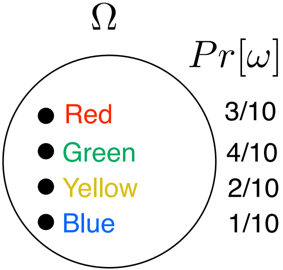

Probability model

$\Omega = \{$Red, Green, Yellow, Blue$\}$

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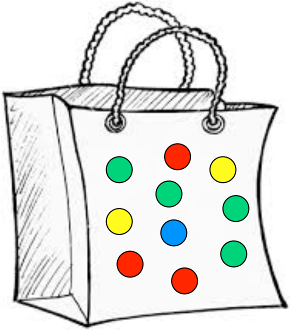

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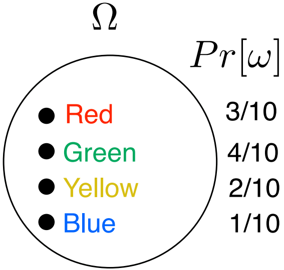

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$\Omega = \{$Red, Green, Yellow, Blue$\}$

$Pr[\text{Red}|\text{Red or Green}] =$

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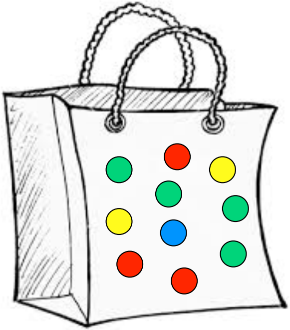

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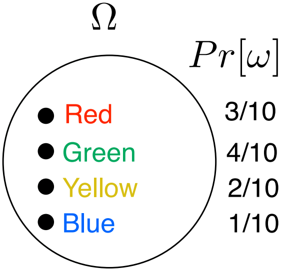

Probability model

$$\Omega = \{\text{Red, Green, Yellow, Blue}\}$$

$$Pr[\text{Red}|\text{Red or Green}] = \frac{3}{7} =$$

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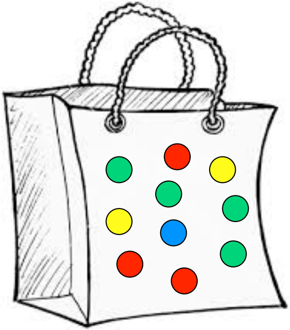

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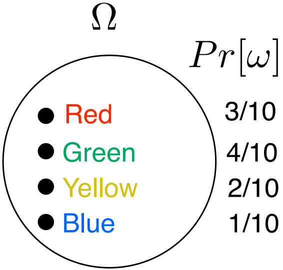

Probability model

$$\Omega = \{\text{Red, Green, Yellow, Blue}\}$$

$$Pr[\text{Red}|\text{Red or Green}] = \frac{3}{7} = \frac{Pr[\text{Red} \cap (\text{Red or Green})]}{Pr[\text{Red or Green}]}$$

Another non-uniform example

Consider $\Omega = \{1, 2, \ldots, N\}$ with $Pr[n] = p_n$.

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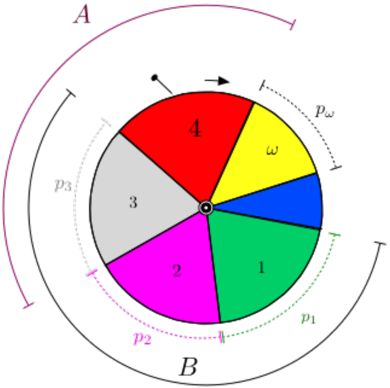

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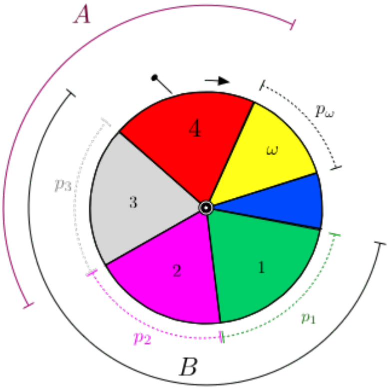

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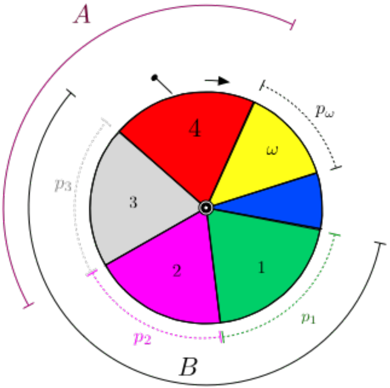

$$Pr[A|B] = \frac{p_3}{p_1 + p_2 + p_3} = \frac{Pr[A \cap B]}{Pr[B]}.$$

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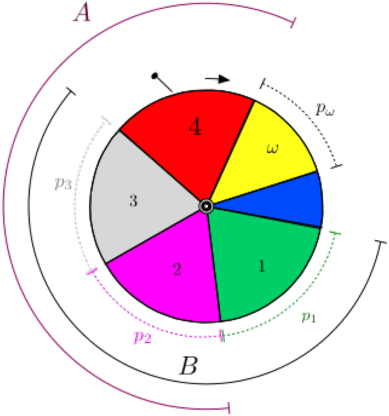

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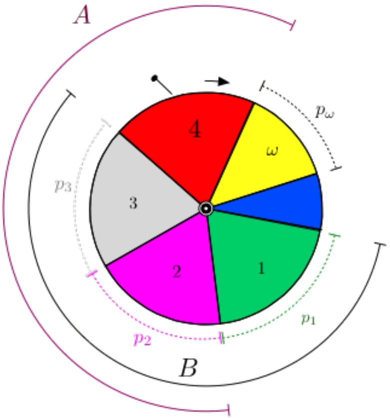

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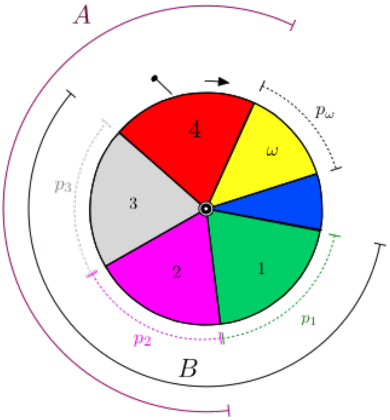

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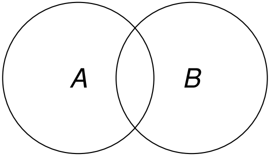
Conditional Probability.

Definition: The **conditional probability** of B given A is

$$Pr[B|A] = \frac{Pr[A \cap B]}{Pr[A]}$$



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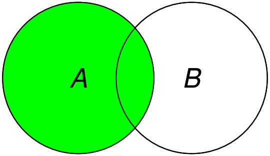

In A!

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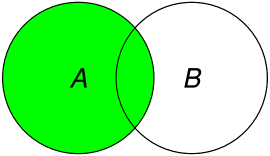

In A!
In B?

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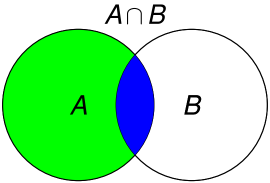

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Must be in $A \cap B$.

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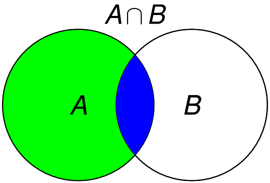

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More fun with conditional probability.

Toss a red and a blue die, sum is 4,

More fun with conditional probability.

Toss a red and a blue die, sum is 4,
What is probability that red is 1?

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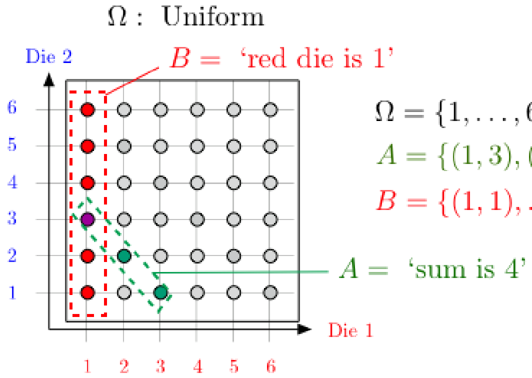

$\Omega = \{1, \ldots, 6\}^2$

$A = \{(1,3), (2,2), (3,1)\}$

$B = \{(1,1), \ldots, (1,6)\}$

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Toss a red and a blue die, sum is 4,
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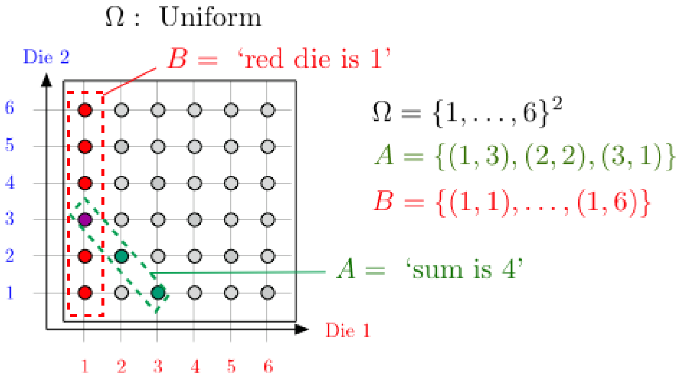

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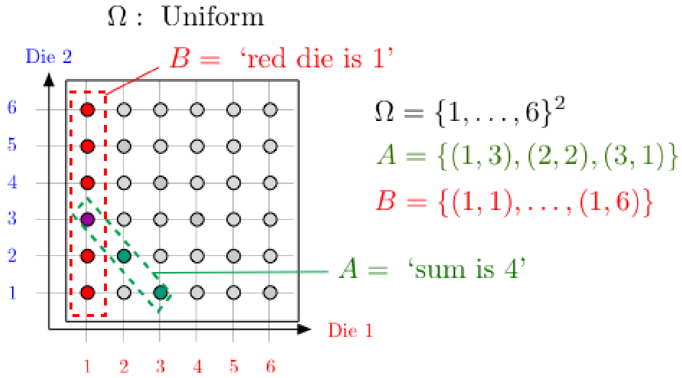

$Pr[B|A] = \frac{|B \cap A|}{|A|} = \frac{1}{3}$; versus $Pr[B] = 1/6$.

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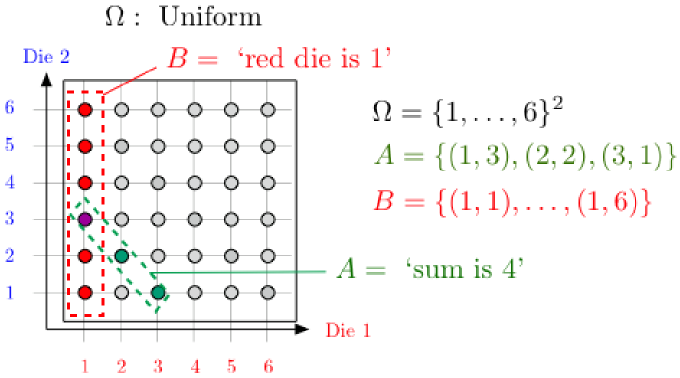

$Pr[B|A] = \frac{|B \cap A|}{|A|} = \frac{1}{3}$; versus $Pr[B] = 1/6$.

B is more likely given *A*.

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Toss a red and a blue die, sum is 7,
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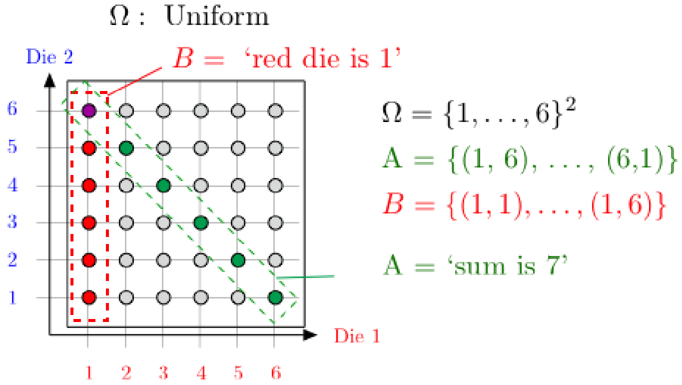

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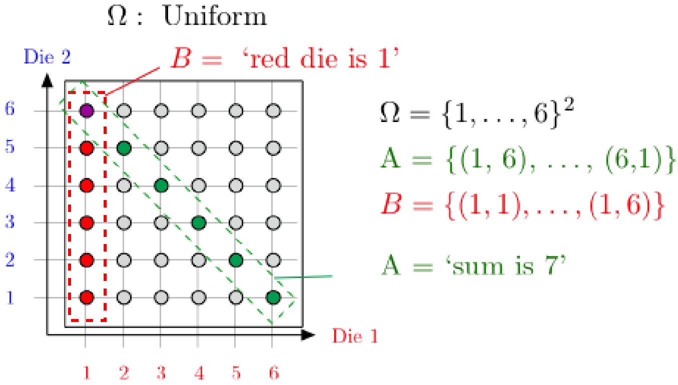

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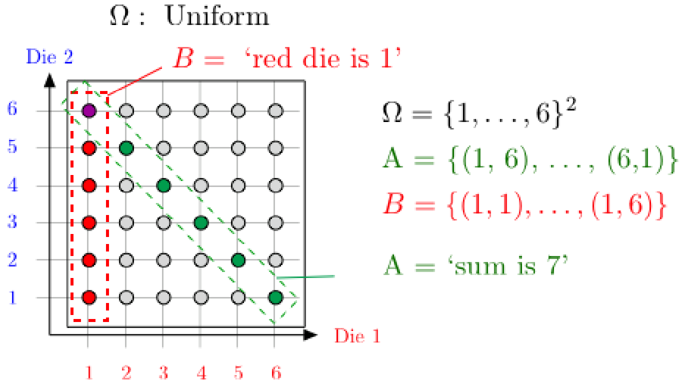

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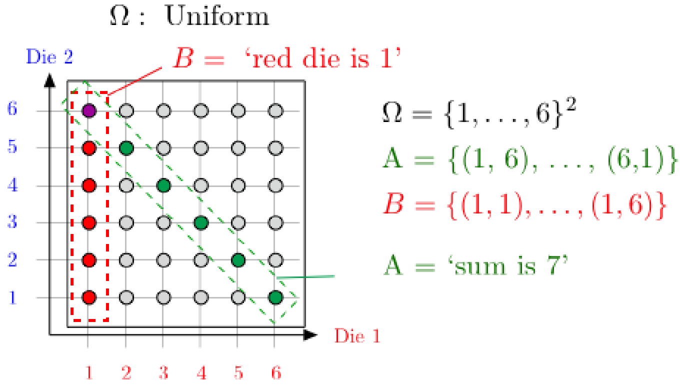

$Pr[B|A] = \frac{|B \cap A|}{|A|} = \frac{1}{6}$; versus $Pr[B] = \frac{1}{6}$.

Observing *A* does not change your mind about the likelihood of *B*.

Emptiness..

Suppose I toss 3 balls into 3 bins.

Emptiness..

Suppose I toss 3 balls into 3 bins.
A =“1st bin empty”;

Emptiness..

Suppose I toss 3 balls into 3 bins.
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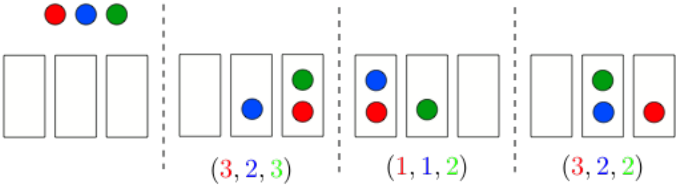
$$\Omega = \{1,2,3\}^3$$



$$\omega = (\text{bin of red ball}, \text{bin of blue ball}, \text{bin of green ball})$$

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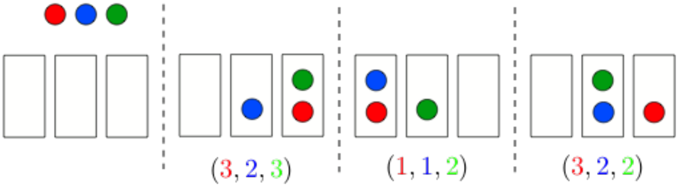

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$Pr[B]$

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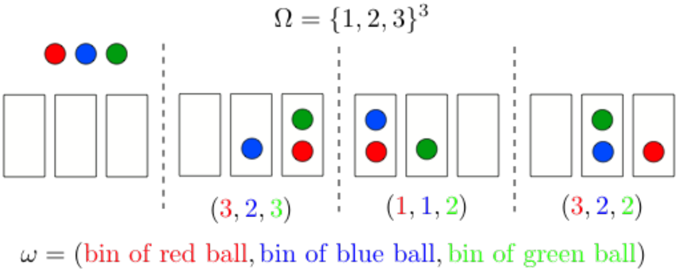

$Pr[B] = Pr[\{(a,b,c) \mid a,b,c \in \{1,3\}] =$

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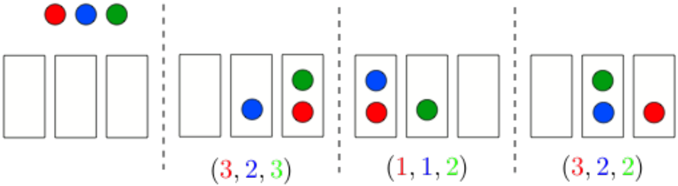

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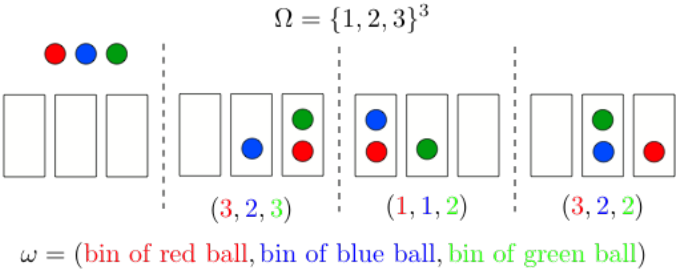

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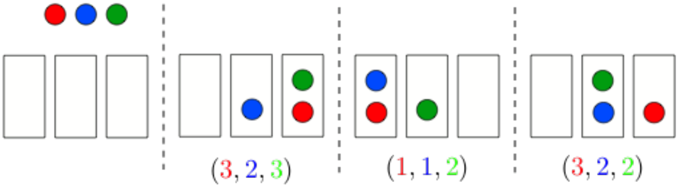

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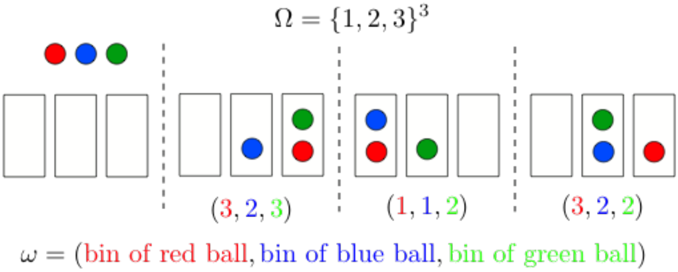

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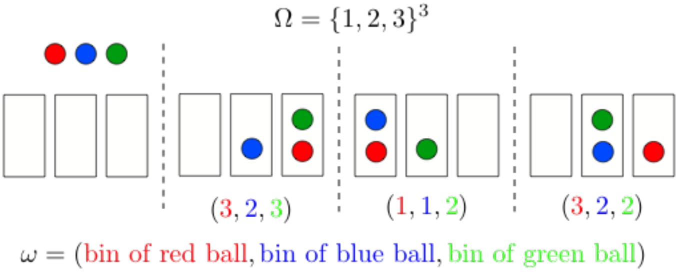

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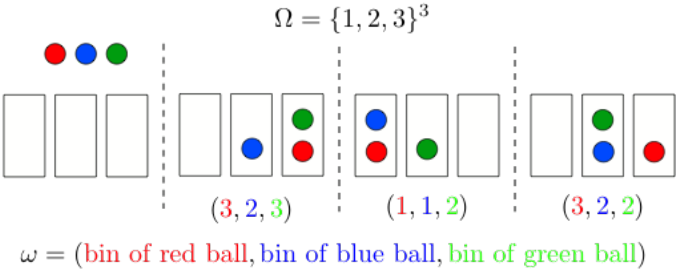

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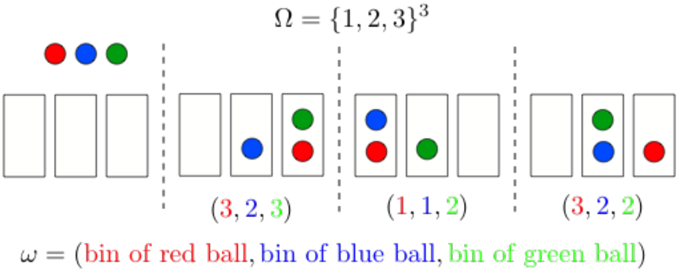

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$Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]}$

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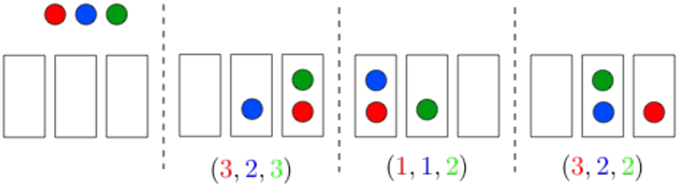

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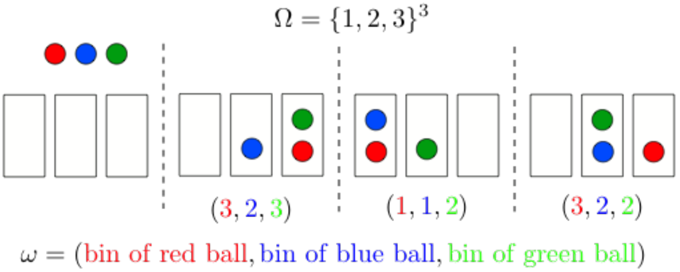

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$Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]} = \frac{(1/27)}{(8/27)} = 1/8$; vs. $Pr[A] = \frac{8}{27}$.

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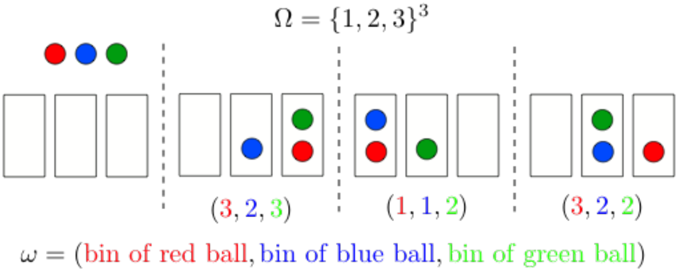
A ="1st bin empty"; B ="2nd bin empty." What is $Pr[A|B]$?



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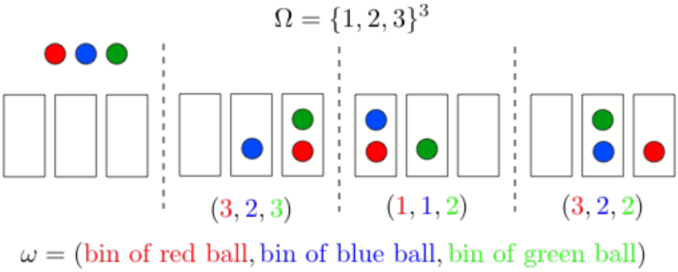

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A is less likely given B: If second bin is empty the first is more likely to have balls in it.

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Flip a fair coin 51 times.

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The likelihood of 51st heads does not depend on the previous flips.

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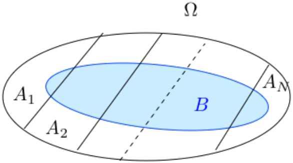
More about such questions later. For fun, check “N. Taleb: Fooled by randomness.”

Total probability

Assume that Ω is the union of the disjoint sets $A_1, \ldots, A_N$.



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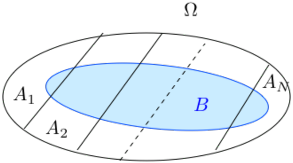

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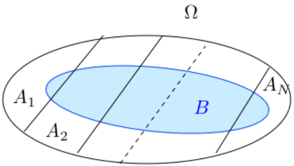

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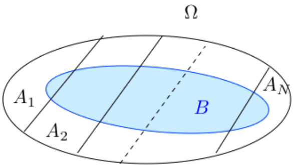

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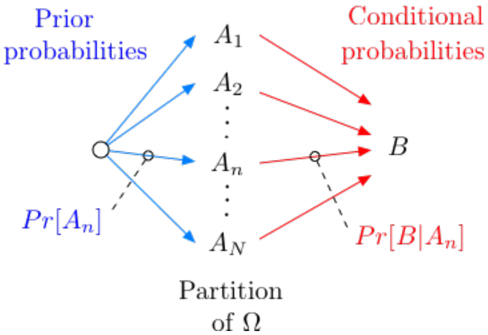

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$$Pr[A|B] = \frac{Pr[A]Pr[B|A]}{Pr[B]} = \frac{(1/2)(1/2)}{(1/2)(1/2) + (1/2)0.6} \approx 0.45.$$

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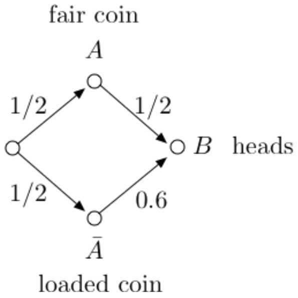

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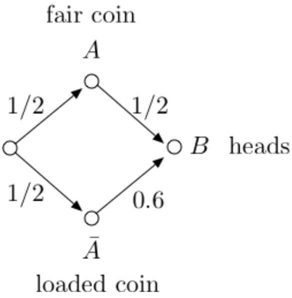

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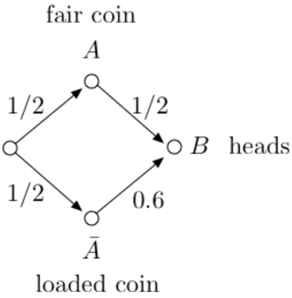

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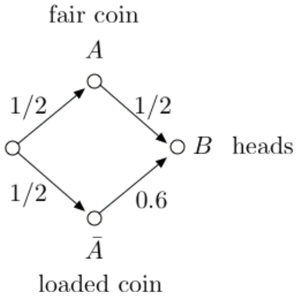

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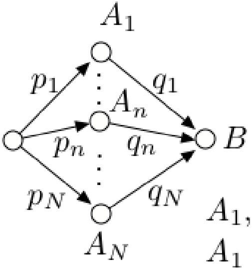

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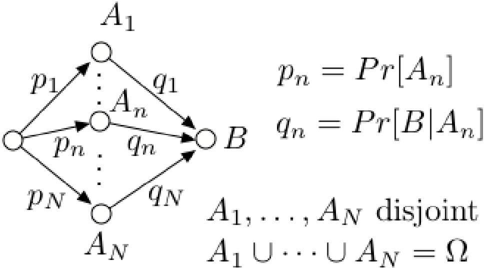

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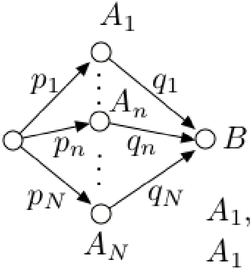

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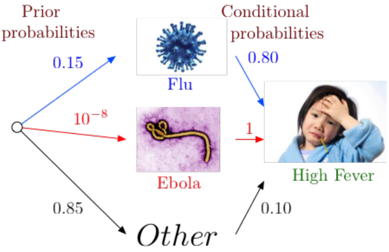
Why do you have a fever?



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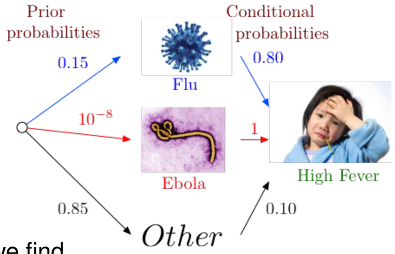

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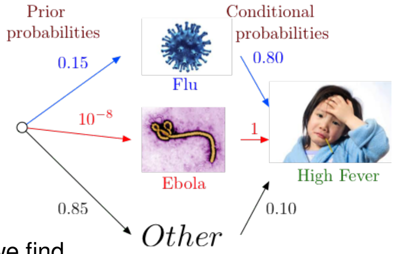

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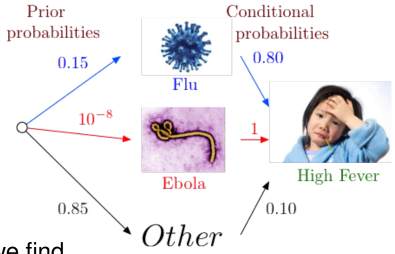

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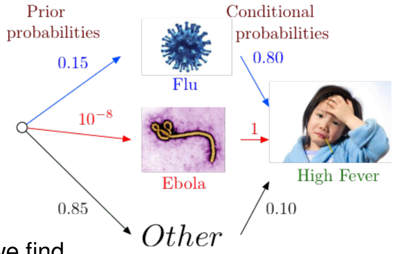

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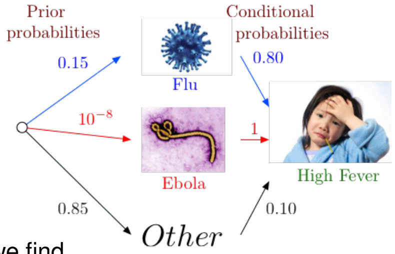

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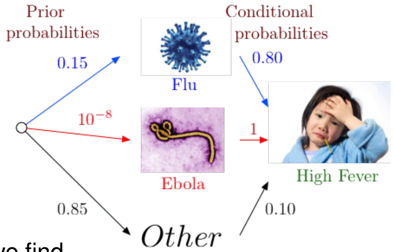

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Bayes' Rule Operations

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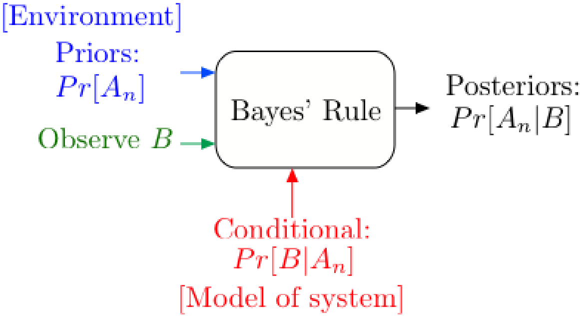

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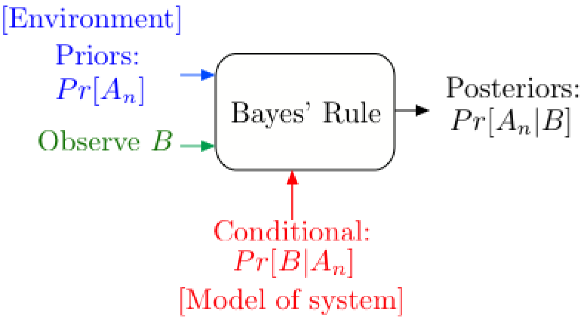

Bayes' Rule is the canonical example of how information changes our opinions.

Thomas Bayes

Thomas Bayes

Portrait used of Bayes in a 1936 book,[1] but it is doubtful whether the portrait is actually of him.[2] No earlier portrait or claimed portrait survives.

Born	c. 1701 London, England
Died	7 April 1761 (aged 59) Tunbridge Wells, Kent, England
Residence	Tunbridge Wells, Kent, England
Nationality	English
Known for	Bayes' theorem

Source: Wikipedia.

Thomas Bayes

A

Fig. 3. *Joshua Bayes (1671–1746).*

$$\xrightarrow{Pr[B|A]}$$

Max over B

B

Thomas Bayes?

A Bayesian picture of Thomas Bayes.

Testing for disease.

Let’s watch TV!!

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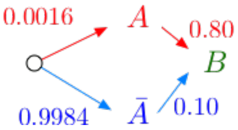

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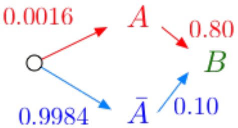

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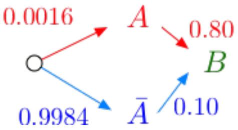

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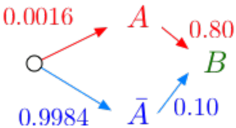

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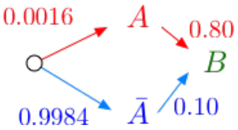

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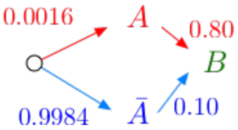

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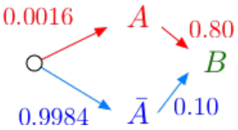

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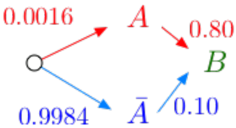

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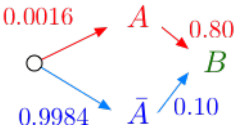
Incontinence..

Bayes Rule.



Using Bayes' rule, we find

$$P[A|B] = \frac{0.0016 \times 0.80}{0.0016 \times 0.80 + 0.9984 \times 0.10} = .013.$$

A 1.3% chance of prostate cancer with a positive PSA test.

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Death.

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Events, Conditional Probability, Independence, Bayes’ Rule

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- All these are possible:

  $Pr[A|B] < Pr[A]; Pr[A|B] > Pr[A]; Pr[A|B] = Pr[A]$.