Today.

More Counting.

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Probability.

Sampling and counting.

First rule: $n_1 \times n_2 \cdots \times n_3$.

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k Samples with replacement from n items: n^k.

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Sample without replacement: $\frac{n!}{(n-k)!}$

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Second rule: when order doesn't matter divide..when possible.

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Sample without replacement and order doesn't matter: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.
"n choose k"

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One-to-one rule: equal in number if one-to-one correspondence.

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"n choose k"

One-to-one rule: equal in number if one-to-one correspondence.

Sample with replacement and order doesn't matter: $\binom{k+n-1}{n-1}$.

Balls in bins.

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5 balls into 10 bins

Balls in bins.

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5 balls into 10 bins
5 samples from 10 possibilities with replacement

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  Example: 5 digit phone numbers.
  5 Balls/places choose from 10 bins/digits.

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5 balls into 10 bins
5 samples from 10 possibilities with replacement
Example: 5 digit phone numbers.
5 Balls/places choose from 10 bins/digits.

5 indistinguishable balls into 52 bins only one ball in each bin

Balls in bins.

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5 samples from 52 possibilities without replacement

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5 balls into 10 bins
5 samples from 10 possibilities with replacement
Example: 5 digit phone numbers.
5 Balls/places choose from 10 bins/digits.

5 indistinguishable balls into 52 bins only one ball in each bin
5 samples from 52 possibilities without replacement
Example: Poker hands.
5 balls/cards into 52 bins/possible cards.

Balls in bins.

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5 balls into 10 bins
5 samples from 10 possibilities with replacement
Example: 5 digit phone numbers.
5 Balls/places choose from 10 bins/digits.

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Example: Poker hands.
5 balls/cards into 52 bins/possible cards.

5 indistinguishable balls into 3 bins

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5 balls into 10 bins
5 samples from 10 possibilities with replacement
Example: 5 digit phone numbers.
5 Balls/places choose from 10 bins/digits.

5 indistinguishable balls into 52 bins only one ball in each bin
5 samples from 52 possibilities without replacement
Example: Poker hands.
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5 indistinguishable balls into 3 bins
5 samples from 3 possibilities with replacement and no order

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Example: 5 digit phone numbers.
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Example: Poker hands.
5 balls/cards into 52 bins/possible cards.

5 indistinguishable balls into 3 bins
5 samples from 3 possibilities with replacement and no order
Dividing 5 dollars among Alice, Bob and Eve.
5 dollars/balls choose from 3 people/bins.

Sum Rule

Two indistinguishable jokers in 54 card deck.
How many 5 card poker hands?

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$\binom{52}{5}$

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$$\binom{52}{5} + \binom{52}{4}$$

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Two distinguishable jokers in 54 card deck.
How many 5 card poker hands? Choose 4 cards plus one of 2 jokers!

$$\binom{52}{5} + 2 * \binom{52}{4} +$$

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Theorem: $\binom{54}{5}$

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Algebraic Proof:

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Algebraic Proof: No need! Above is combinatorial proof.

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Combinatorial Proofs.

Theorem: $\binom{n}{k} = \binom{n}{n-k}$

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Theorem: $\binom{n}{k} = \binom{n}{n-k}$

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How many subsets of size k?
Choose a subset of size $n-k$
and what's left out

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How many subsets of size k?
Choose a subset of size $n-k$
and what's left out is a subset of size k.

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Proof: How many subsets of size k? $\binom{n}{k}$

How many subsets of size k?
Choose a subset of size $n-k$
and what's left out is a subset of size k.
Choosing a subset of size k is same

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How many subsets of size k?
Choose a subset of size $n-k$
and what's left out is a subset of size k.
Choosing a subset of size k is same
as choosing $n-k$ elements to not take.

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$\implies \binom{n}{n-k}$ subsets of size k.

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$\square$

Pascal's Triangle

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0

1 1

Pascal's Triangle

0

1 1

1 2 1

Pascal's Triangle

			0			
		1		1		
	1		2		1	
1		3		3		1

Pascal's Triangle

0

1 1

1 2 1

1 3 3 1

1 4 6 4 1

Pascal's Triangle

				0				
			1		1			
		1		2		1		
	1		3		3		1	
1		4		6		4		1

Pascal's Triangle

0

1 1

1 2 1

1 3 3 1

1 4 6 4 1

Row n: coefficients of $(1+x)^n = (1+x)(1+x)\cdots(1+x)$.

Pascal's Triangle

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

Row n: coefficients of $(1+x)^n = (1+x)(1+x)\cdots(1+x)$.

Foil (4 terms)

Pascal's Triangle

0

1 1

1 2 1

1 3 3 1

1 4 6 4 1

Row n: coefficients of $(1+x)^n = (1+x)(1+x)\cdots(1+x)$.

Foil (4 terms) on steroids:

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$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
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$$

Row n: coefficients of $(1+x)^n = (1+x)(1+x)\cdots(1+x)$.

Foil (4 terms) on steroids:
2^n terms:

Pascal's Triangle

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

Row n: coefficients of $(1+x)^n = (1+x)(1+x)\cdots(1+x)$.

Foil (4 terms) on steroids:

2^n terms: choose 1 or x froom each factor of $(1+x)$.

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$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
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$$\begin{array}{ccccccccc} &&&& 0 &&&& \\ &&& 1 && 1 &&& \\ && 1 && 2 && 1 && \\ & 1 && 3 && 3 && 1 & \\ 1 && 4 && 6 && 4 && 1 \end{array}$$

Row n: coefficients of $(1+x)^n = (1+x)(1+x)\cdots(1+x)$.

Foil (4 terms) on steroids:
2^n terms: choose 1 or x froom each factor of $(1+x)$.

Simplify: collect all terms corresponding to x^k.

Pascal's Triangle

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

Row n: coefficients of $(1+x)^n = (1+x)(1+x)\cdots(1+x)$.

Foil (4 terms) on steroids:
  2^n terms: choose 1 or x froom each factor of $(1+x)$.

Simplify: collect all terms corresponding to x^k.
  Coefficient of x^k is $\binom{n}{k}$: choose k factors where x is in product.

Pascal's Triangle

$$\begin{matrix} & & & & 0 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{matrix}$$

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$$\begin{matrix} & \binom{0}{0} & \\ \binom{1}{0} & & \binom{1}{1} \end{matrix}$$

Pascal's Triangle

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0
1 1
1 2 1
1 3 3 1
1 4 6 4 1

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$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

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 & & & & 0 & & & & \\
 & & & 1 & & 1 & & & \\
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\end{array}
$$

Pascal's rule $\implies \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Combinatorial Proofs.

Theorem: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Proof: How many size k subsets of $n+1$?

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How many size k subsets of $n+1$?
How many contain the first element?

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How many size k subsets of $n+1$?
How many contain the first element?
Choose first element,
need to choose $k-1$ more from remaining n elements.

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$\implies \binom{n}{k-1}$

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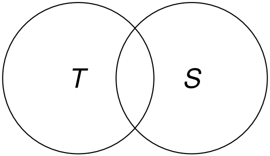

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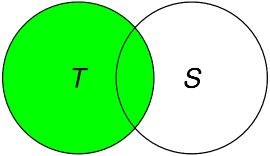

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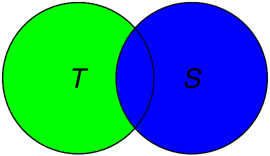

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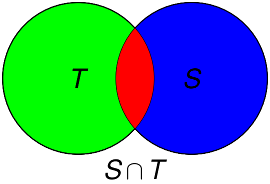

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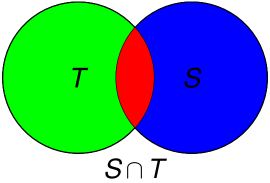

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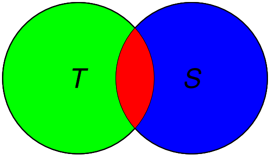

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Answer: $|S| + |T| - |S \cap T| = 10^9 + 10^9 - 10^8$.

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Disjoint – so add!

CS70: On to probability.

Modeling Uncertainty: Probability Space

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Modeling Uncertainty: Probability Space

1. Key Points
2. Random Experiments
3. Probability Space

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H

T

- Possible outcomes:

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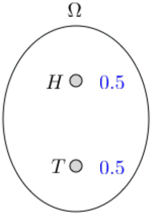
Physical Experiment



Probability Model

Random Experiment: Flip one Fair Coin

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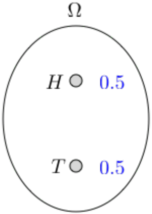

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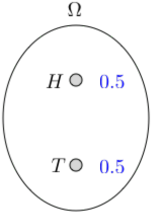

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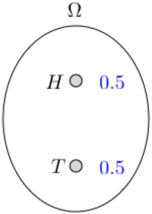

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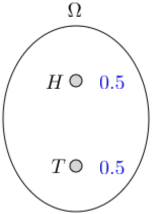

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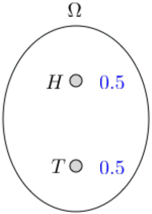

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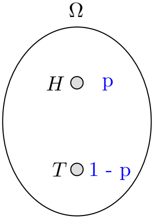
Flip an unfair (biased, loaded) coin: model

Physical Experiment



Probability Model

Flip Two Fair Coins

Flip Two Fair Coins

- Possible outcomes:

Flip Two Fair Coins

- Possible outcomes: $\{HH, HT, TH, TT\}$

Flip Two Fair Coins

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Flip Glued Coins

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Flips two coins glued together side by side:

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Glued coins

50%

50%

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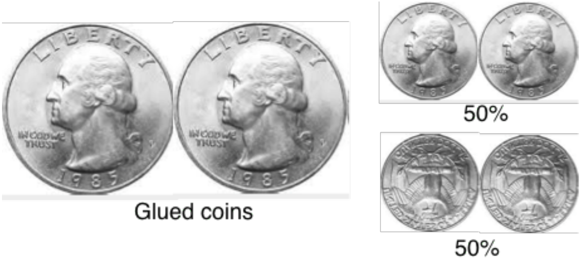

- Possible outcomes: $\{HH, TT\}$.
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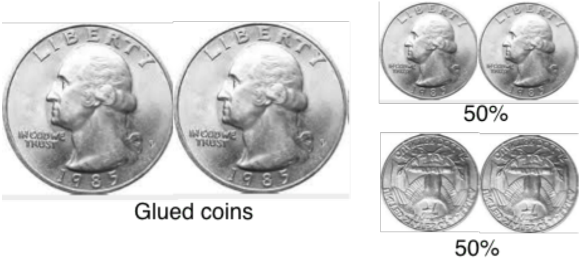

- Possible outcomes: $\{HH, TT\}$.
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- Note: Coins are glued so that they show the same face.

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- Possible outcomes: $\{HT, TH\}$.
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50%

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- Possible outcomes: $\{HT, TH\}$.
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Flip Glued Coins

Flips two coins glued together side by side:

- Possible outcomes: $\{HT, TH\}$.
- Likelihoods: $HT : 0.5, TH : 0.5$.
- Note: Coins are glued so that they show different faces.

Flip two Attached Coins

Flip two Attached Coins

Flips two coins attached by a spring:

Flip two Attached Coins

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- Possible outcomes: $\{HH, HT, TH, TT\}$.
- Likelihoods: $HH : 0.4, HT : 0.1, TH : 0.1, TT : 0.4$.

Flip two Attached Coins

Flips two coins attached by a spring:

- Possible outcomes: $\{HH, HT, TH, TT\}$.
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Flipping Two Coins

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Here is a way to summarize the four random experiments:

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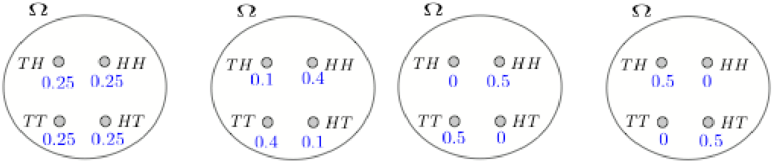

[1] [2] [3] [4]

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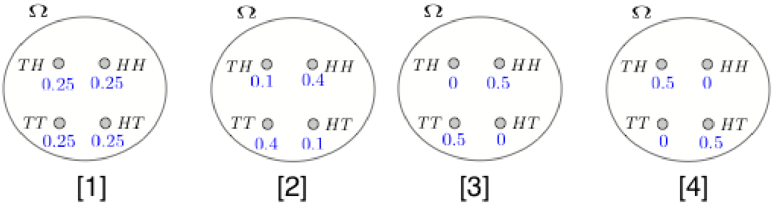

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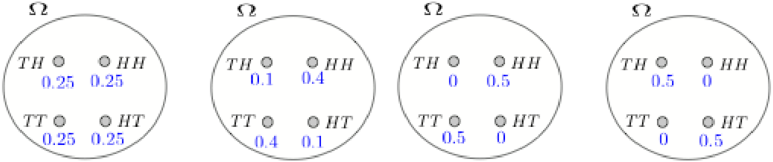

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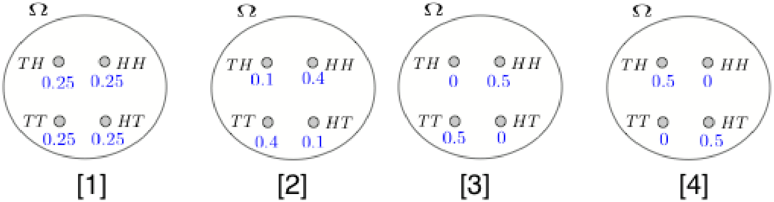

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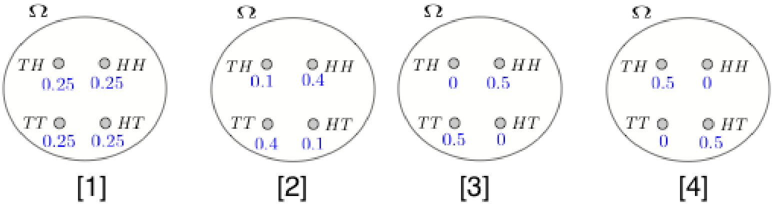

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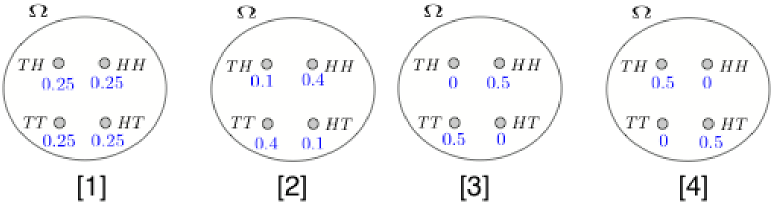

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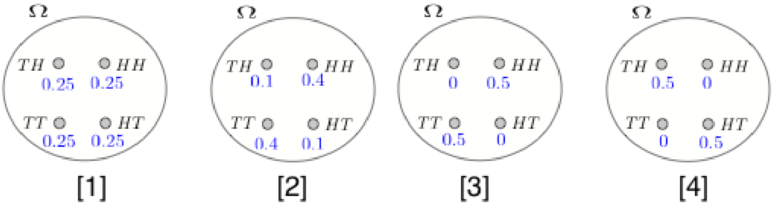

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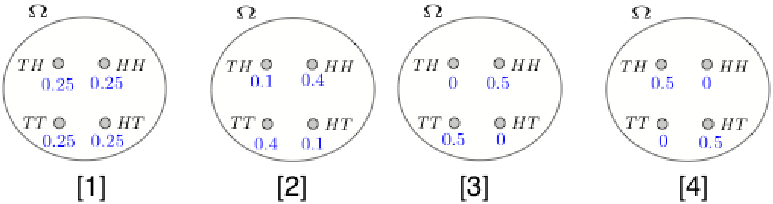

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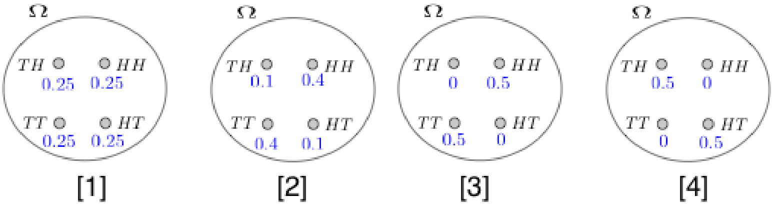

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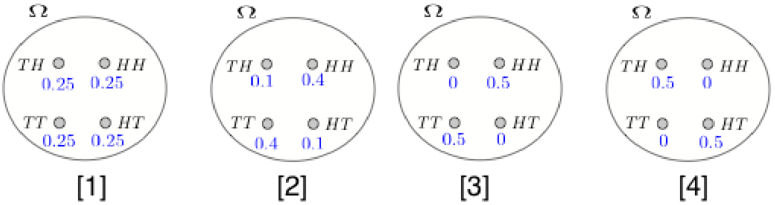

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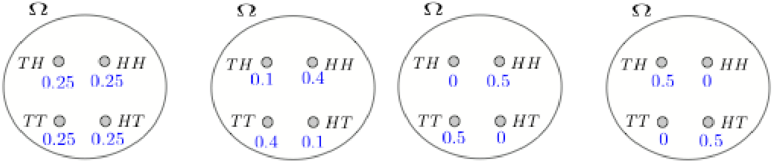

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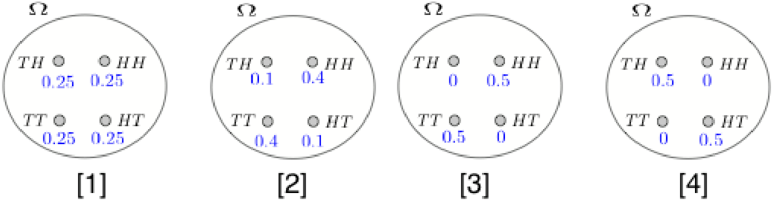

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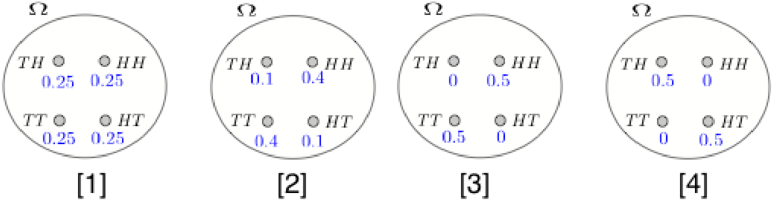

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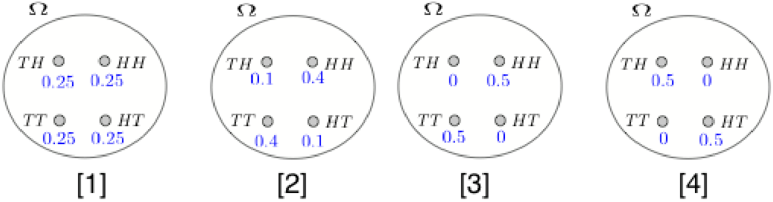

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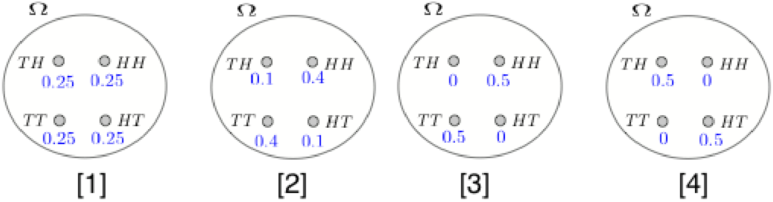

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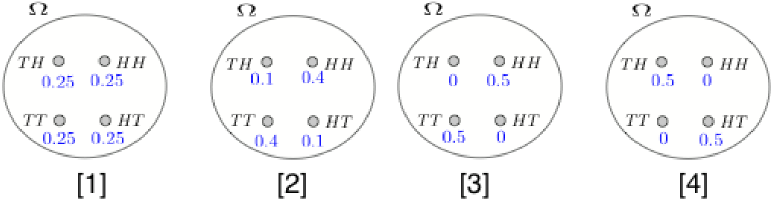

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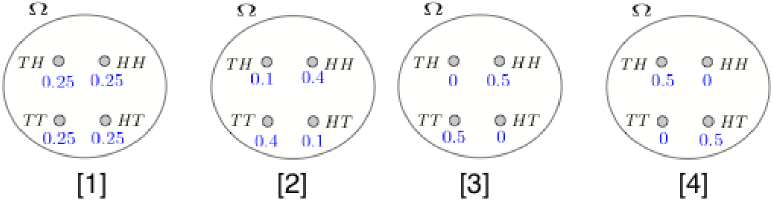

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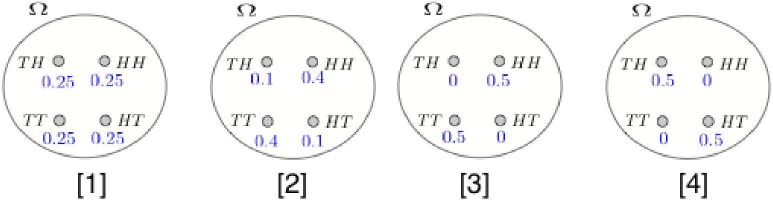

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- Ω and the probabilities specify the random experiment.

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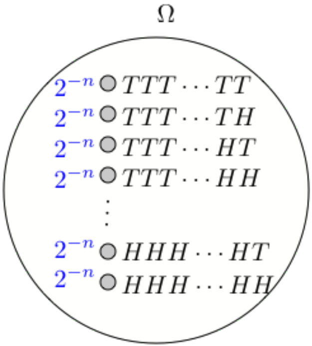

Roll two Dice

Roll a balanced 6-sided die twice:

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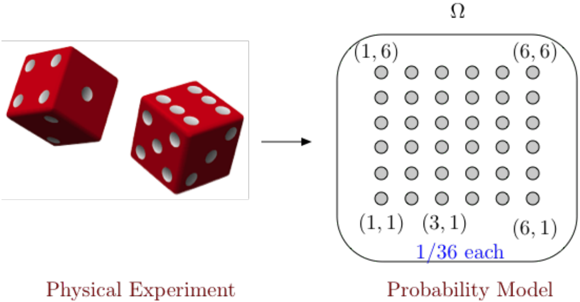
- Likelihoods: $1/36$ for each.



Probability Space.

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3. Assign a probability to each outcome: $Pr : \Omega \to [0,1]$.

   (a) $Pr[H] = p, Pr[T] = 1 - p$ for some $p \in [0,1]$

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   (c) $Pr[\ \underline{A\spadesuit\ A\diamondsuit\ A\clubsuit\ A\heartsuit\ K\spadesuit}\ ] = \cdots = 1/\binom{52}{5}$

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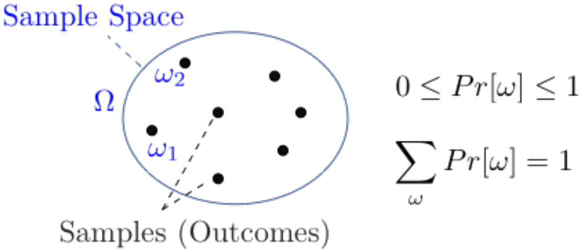

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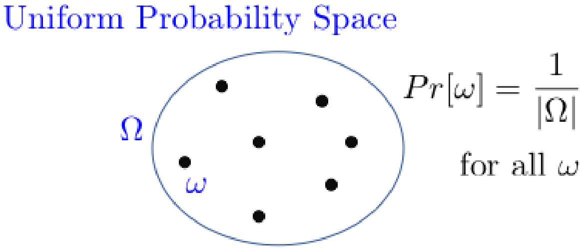

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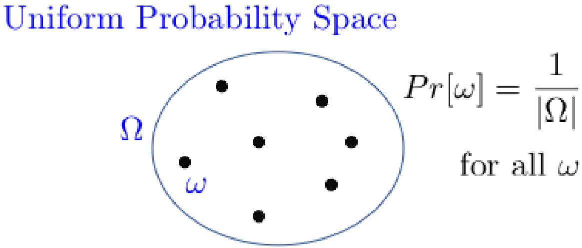

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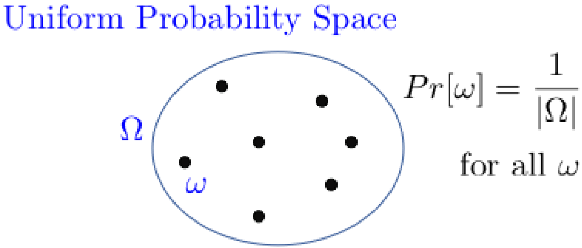


Examples:

- Flipping two fair coins, dealing a poker hand are uniform probability spaces.
- Flipping a biased coin is not a uniform probability space.

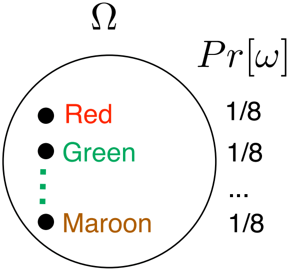
Probability Space: Formalism

Simplest physical model of a uniform probability space:

Probability Space: Formalism

Simplest physical model of a uniform probability space:

Physical experiment



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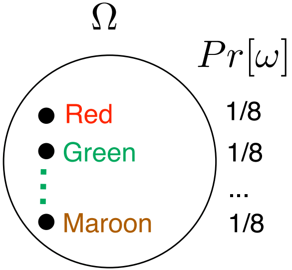

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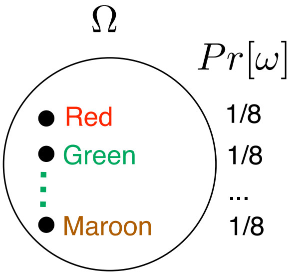

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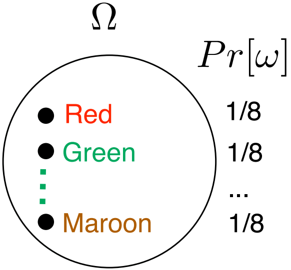

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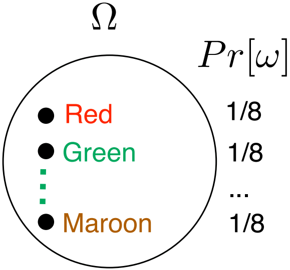

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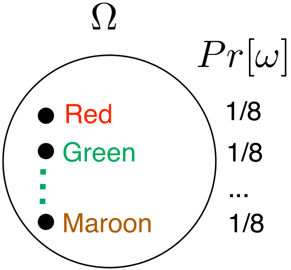

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$$Pr[\text{blue}] = \frac{1}{8}.$$

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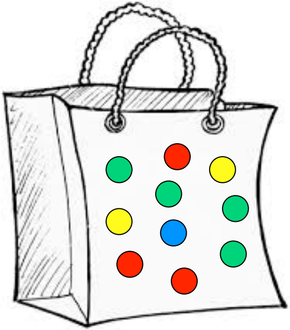

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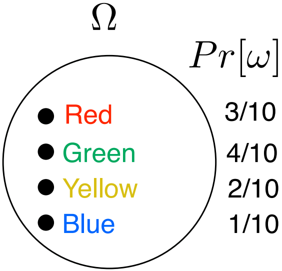

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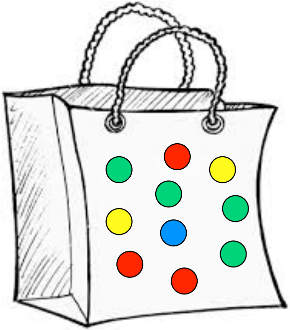

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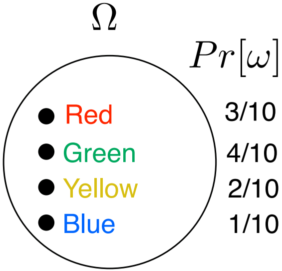

Probability model

$\Omega = \{$Red, Green, Yellow, Blue$\}$

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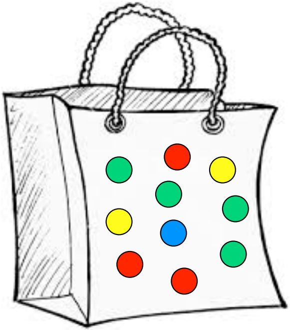

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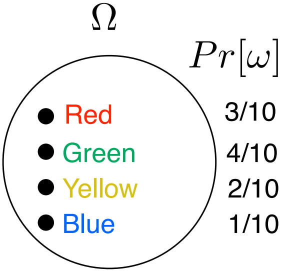

Probability model

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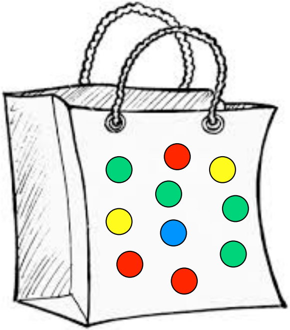

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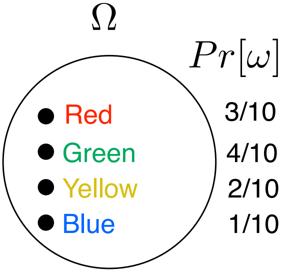

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$$\Omega = \{\text{Red, Green, Yellow, Blue}\}$$

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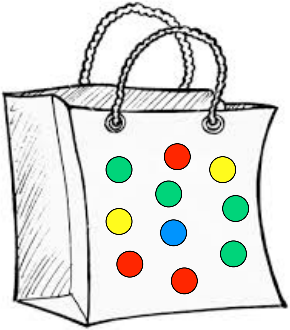

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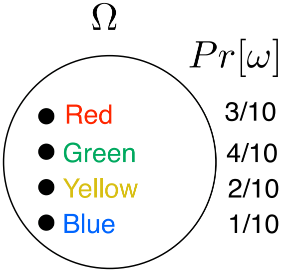

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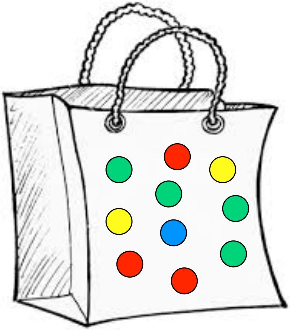

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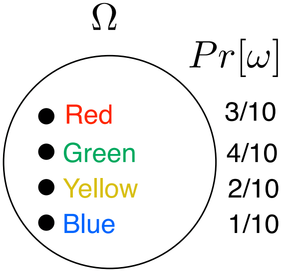

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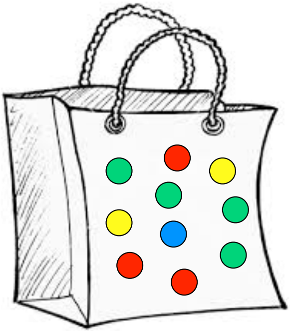

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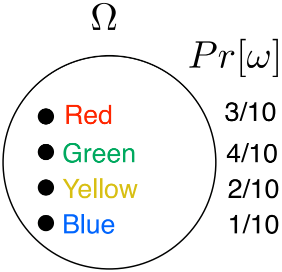

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Note: Probabilities are restricted to rational numbers: $\frac{N_k}{N}$.

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Physical model of a general non-uniform probability space:

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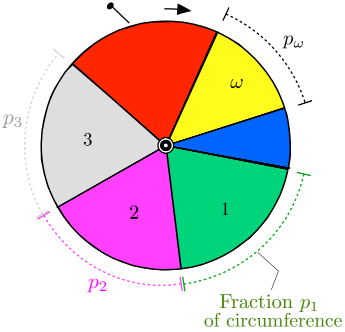

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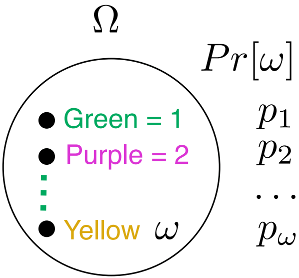

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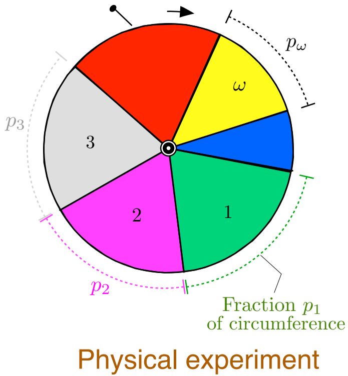

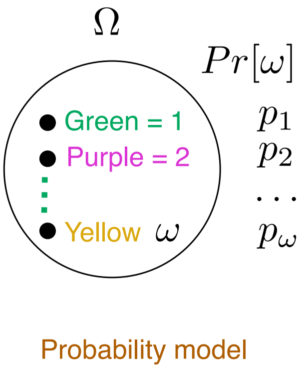

The roulette wheel stops in sector ω with probability p_ω.

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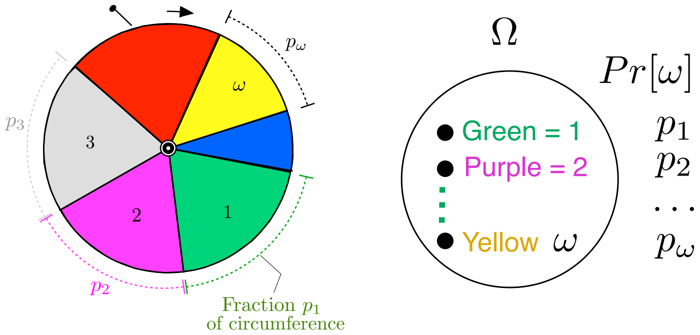

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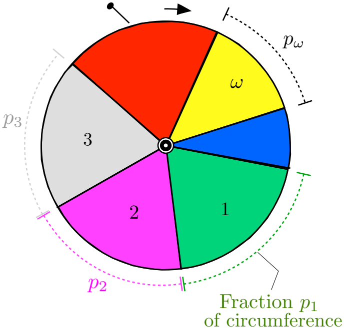

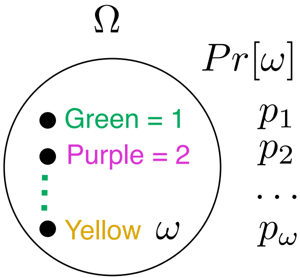

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Lecture 15: Summary

Modeling Uncertainty: Probability Space

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1. Random Experiment

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