Today.

Wrapup of Polynomials.

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..and modular arithmetic.

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Coutability and Uncountability.

Reed-Solomon code.

Problem: Communicate n packets $m_1, \ldots, m_n$
on noisy channel that corrupts $\leq k$ packets.

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Solve.

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Solve. Then output $P(x) = Q(x)/E(x)$.

Berlekamp-Welsh algorithm decodes correctly when k errors!

Summary: polynomials.

Set of $d+1$ points determines degree d polynomial.

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Other Algebraic-Geometric codes.

Wrapping up: RSA example with “easy” extended gcd.

Example: $p = 7$, $q = 11$.

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$$7(0) + 60(1) = 60$$

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$$\begin{aligned} 7(0) + 60(1) &= 60 \\ 7(1) + 60(0) &= 7 \end{aligned}$$

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Choose $e = 7$, since $\gcd(7,60) = 1$.
egcd(7,60).

$$\begin{aligned} 7(0) + 60(1) &= 60 \\ 7(1) + 60(0) &= 7 \\ 7(-8) + 60(1) &= 4 \end{aligned}$$

Wrapping up: RSA example with "easy" extended gcd.

Example: $p = 7$, $q = 11$.

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Confirm: $-119 + 120 = 1$

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Confirm: $-119 + 120 = 1$

$d = e^{-1} = -17 = 43 = \pmod{60}$

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Modular arithmetic modulo a prime.

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Add, subtract, commutative, associative,

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Next up: how big is infinity.

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- Countable
- Countably infinite.
- Enumeration

How big are the reals or the integers?

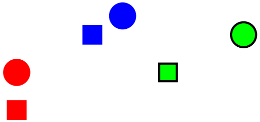
Infinite!

How big are the reals or the integers?

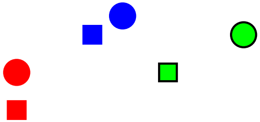
Infinite!

Is one bigger or smaller?

Same size?



Same size?



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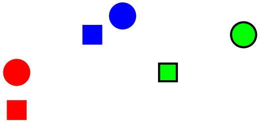
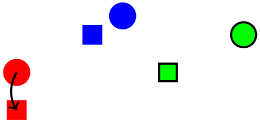


Same number?

Make a function f : Circles $\to$ Squares.

Same size?

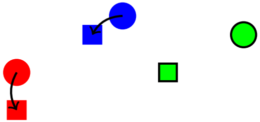


Same number?

Make a function f : Circles $\to$ Squares.

f(red circle) $=$ red square

Same size?



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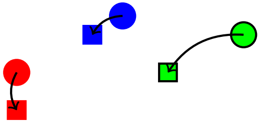


Same number?

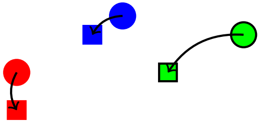
Make a function f : Circles $\rightarrow$ Squares.

f(red circle) $=$ red square

f(blue circle) $=$ blue square

f(circle with black border) $=$ square with black border

Same size?



Same number?

Make a function f : Circles $\rightarrow$ Squares.

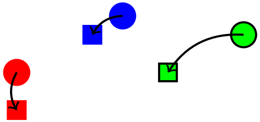
f(red circle) $=$ red square

f(blue circle) $=$ blue square

f(circle with black border) $=$ square with black border

One to one.

Same size?



Same number?

Make a function f : Circles $\rightarrow$ Squares.

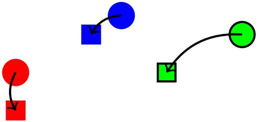
f(red circle) $=$ red square

f(blue circle) $=$ blue square

f(circle with black border) $=$ square with black border

One to one. Each circle mapped to different square.

Same size?



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f(red circle) $=$ red square
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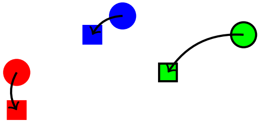
One to One: For all $x, y \in D$, $x \neq y \implies f(x) \neq f(y)$.

Same size?



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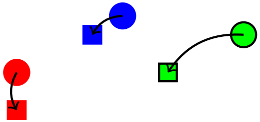

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Onto. Each square mapped to from some circle .

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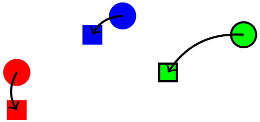

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Onto: For all $s \in R$, $\exists c \in D, s = f(c)$.

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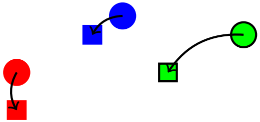

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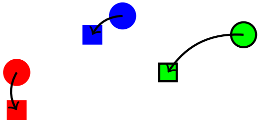

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Isomorphism principle: If there is $f : D \rightarrow R$ that is one to one and onto, then, $|D| = |R|$.

Isomorphism principle.

Given a function, $f : D \rightarrow R$.

Isomorphism principle.

Given a function, $f : D \to R$.
One to One:

Isomorphism principle.

Given a function, $f : D \rightarrow R$.

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Isomorphism principle:

If there is a bijection $f : D \to R$ then $|D| = |R|$.

Countable.

How to count?

Countable.

How to count?

0,

Countable.

How to count?

0, 1,

Countable.

How to count?

0, 1, 2,

Countable.

How to count?

0, 1, 2, 3,

Countable.

How to count?

0, 1, 2, 3, . . .

Countable.

How to count?

0, 1, 2, 3, . . .

The Counting numbers.

Countable.

How to count?

0, 1, 2, 3, . . .

The Counting numbers.
The natural numbers! N

Countable.

How to count?

0, 1, 2, 3, ...

The Counting numbers.
The natural numbers! N

Definition: S is **countable** if there is a bijection between S and some subset of N.

Countable.

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0, 1, 2, 3, ...

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The natural numbers! N

Definition: S is **countable** if there is a bijection between S and some subset of N.

If the subset of N is finite, S has finite **cardinality**.

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0, 1, 2, 3, ...

The Counting numbers.
The natural numbers! N

Definition: S is **countable** if there is a bijection between S and some subset of N.

If the subset of N is finite, S has finite **cardinality**.

If the subset of N is infinite, S is **countably infinite**.

Where’s 0?

Which is bigger?

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Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Where's 0?

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Natural numbers. 0,

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Natural numbers. $0, 1,$

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More natural numbers!

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More natural numbers!

Consider $f(z) = z - 1$.

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For any two $z_1 \neq z_2$

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For any natural number n, for $z = n + 1$,

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Bijection!

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Onto for $\mathbb{N}$

Bijection! $\implies |\mathbb{Z}^+| = |\mathbb{N}|$.

But.. but Where's zero? "Comes from 1."

A bijection is a bijection.

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Notice that there is a bijection between N and Z^+ as well.

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Bijection from A to $B \implies$ a bijection from B to A.

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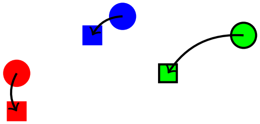

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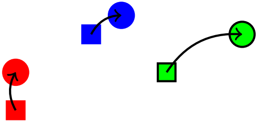

Inverse function!

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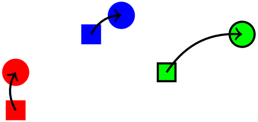

Inverse function!

Can prove equivalence either way.

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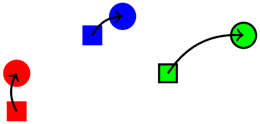

Inverse function!

Can prove equivalence either way.
Bijection to or from natural numbers implies countably infinite.

More large sets.

E - Even natural numbers?

More large sets.

E - Even natural numbers?

$f : N \rightarrow E$.

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E - Even natural numbers?

$f : N \rightarrow E$.

$f(n) \rightarrow 2n$.

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Onto:

More large sets.

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Onto: $\forall e \in E$, $f(e/2) = e$.

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One-to-one: $\forall x, y \in N, x \neq y \implies 2x \neq 2y$.

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Evens are countably infinite.

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One-to-one: $\forall x, y \in N, x \neq y \implies 2x \neq 2y. \equiv f(x) \neq f(y)$

Evens are countably infinite.
Evens are same size as all natural numbers.

All integers?

What about Integers, Z?

All integers?

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Define $f : N \to Z$.

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if n odd.} \end{cases}$$

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if $z \geq 0$, $f(2z) = z$ and $2z \in N$.

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$\cdots$

Onto: For any $z \in Z$,
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if $z < 0$, $f(2|z| - 1) = z$ and $2|z| + 1 \in N$.

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Integers and naturals have same size!

Listings..

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Another View:

n	$f(n)$

Listings..

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Another View:

n	$f(n)$
0	0

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$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if n odd.} \end{cases}$$

Another View:

n	$f(n)$
0	0
1	-1

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Another View:

n	$f(n)$
0	0
1	-1
2	1

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$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if n odd.} \end{cases}$$

Another View:

n	$f(n)$
0	0
1	-1
2	1
3	-2

Listings..

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Another View:

n	$f(n)$
0	0
1	-1
2	1
3	-2
4	2

Listings..

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if n odd.} \end{cases}$$

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61A

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61A —- streams!

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All countably infinite sets have the same cardinality.

Enumeration example.

All binary strings.

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$B = \{0, 1\}^*$.

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$B = \{\phi;, 0, 00, 000, 0000, ...\}$
Never get to 1.

More fractions?

Enumerate the rational numbers in order...

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Can't list in "order".

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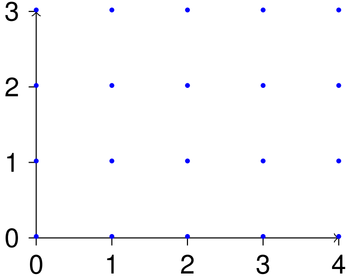

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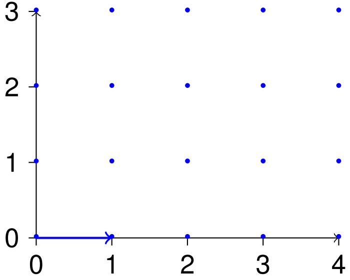

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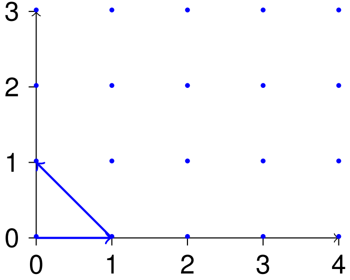

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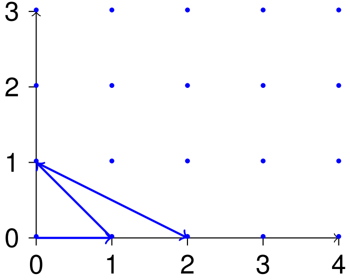

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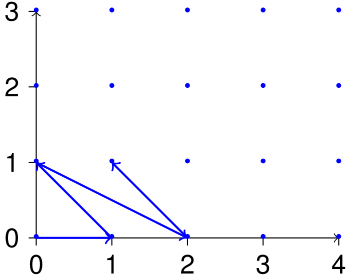

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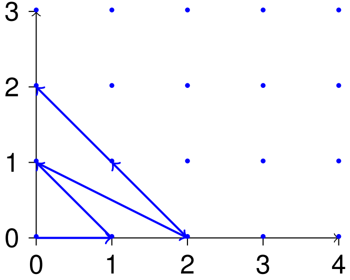

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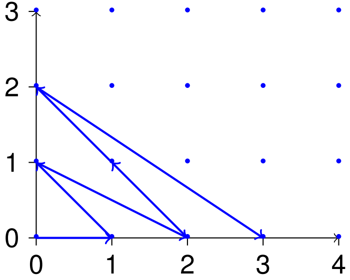

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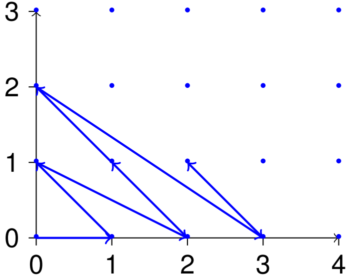

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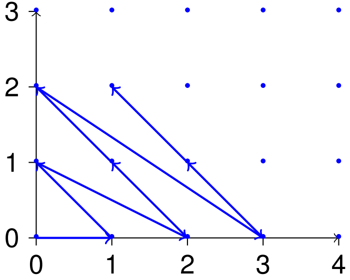

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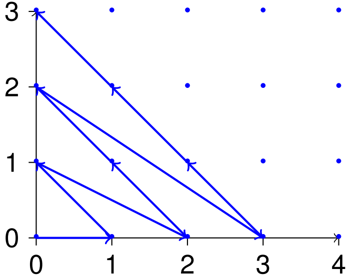

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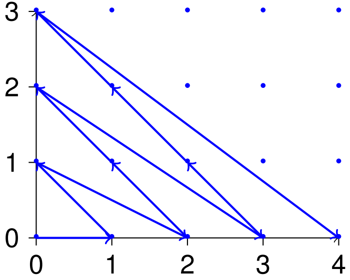

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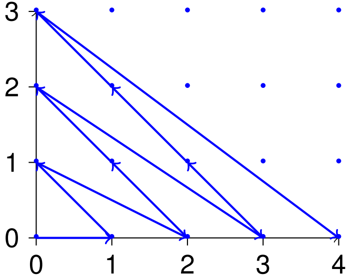

The pair (a,b), is in first $(a+b+1)(a+b)/2$ elements of list!

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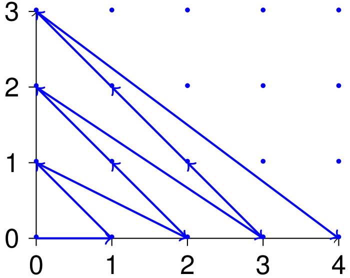

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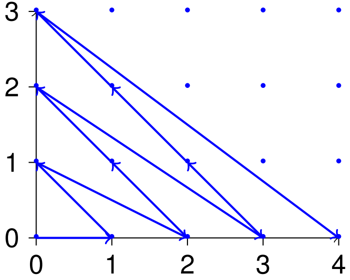

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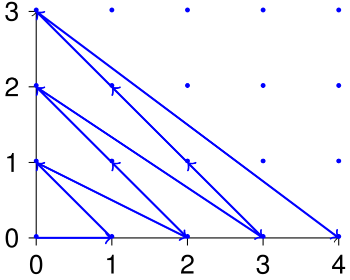

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Same size as the natural numbers!!

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Positive rational number.

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Negative rationals are countable.

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Rationals?

Positive rational number.
Lowest terms: a/b
$a, b \in N$
with $gcd(a,b) = 1$.

Infinite subset of $N \times N$.

Countably infinite!

All rational numbers?

Negative rationals are countable. (Same size as positive rationals.)

Put all rational numbers in a list.

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Real numbers..

Real numbers are same size as integers?

The reals.

Are the set of reals countable?

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Lets consider the reals $[0, 1]$.

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Construct "diagonal" number:

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Construct “diagonal” number: .7

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⋮

Construct "diagonal" number: .77

Diagonalization.

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Construct “diagonal” number: .7767

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Contradiction!

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Contradiction!

Subset $[0,1]$ is not countable!!

All reals?

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Subset $[0, 1]$ is not countable!!

What about all reals?

All reals?

Subset $[0, 1]$ is not countable!!

What about all reals?
No.

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Any subset of a countable set is countable.

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Any subset of a countable set is countable.

If reals are countable then so is $[0, 1]$.

Diagonalization.

1. Assume that a set S can be enumerated.

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   $\implies t$ is not in the list.

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6. Contradiction.

Another diagonalization.

The set of all subsets of N.

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Example subsets of N: $\{0\}$,

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$, $\{0,\ldots,7\}$,

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Example subsets of *N*: $\{0\}$, $\{0, \ldots, 7\}$,
evens, odds,

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Theorem: The set of all subsets of N is not countable.

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L does not contain all subsets of N.

Contradiction.

Theorem: The set of all subsets of N is not countable.
(The set of all subsets of S, is the **powerset** of N.)

Diagonalize Natural Number.

Natural numbers have a listing, *L*.

Diagonalize Natural Number.

Natural numbers have a listing, L.

Make a diagonal number, D:
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Any natural number has a finite number of digits.

“Construction” requires an infinite number of digits.

The Continuum hypothesis.

There is no set with cardinality between the naturals and the reals.

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There is no set with cardinality between the naturals and the reals.

First of Hilbert's problems!

Cardinalities of uncountable sets?

Cardinality of $[0, 1]$ smaller than all the reals?

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$f : R^+ \rightarrow [0, 1]$.

Cardinalities of uncountable sets?

Cardinality of $[0,1]$ smaller than all the reals?

$f : R^+ \rightarrow [0,1]$.

$$f(x) = \begin{cases} x + \frac{1}{2} & 0 \leq x \leq 1/2 \\ \frac{1}{4x} & x > 1/2 \end{cases}$$

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If both in $[0,1/2]$, a shift $\implies f(x) \neq f(y)$.
If neither in $[0,1/2]$

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If one is in $[0, 1/2]$ and one isn't,

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If both in $[0,1/2]$, a shift $\implies f(x) \neq f(y)$.
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$[0,1]$ is same cardinality as nonnegative reals!

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The powerset of a set is the set of all subsets.

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Self reference.

More on...

...Tuesday..

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