CS 70 Discrete Mathematics and Probability Theory
Spring 2017 Rao

HW 7

1 Sundry

Before you start your homework, write down your team. Who else did you work with on this homework? List names and email addresses. (In case of homework party, you can also just describe the group.) How did you work on this homework? Working in groups of 3-5 will earn credit for your "Sundry" grade.

Please copy the following statement and sign next to it:

I certify that all solutions are entirely in my words and that I have not looked at another student's solutions. I have credited all external sources in this write up.

2 More Countability

Given:

- A is a countable set, non-empty set. For all $i \in A$, S_i is an uncountable set.
- B is an uncountable set. For all $i \in B$, Q_i is a countable set.

For each of the following, decide if the expression is "Always Countable", "Always Uncountable", "Sometimes Countable, Sometimes Uncountable".

For the "Always" cases, prove your claim. For the "Sometimes" case, provide two examples – one where the expression is countable, and one where the expression is uncountable.

(a) $\bigcup_{i \in A} S_i$

(b) $\bigcap_{i \in A} S_i$

(c) $\bigcup_{i \in B} Q_i$

(d) $\bigcap_{i \in B} Q_i$

(e) $A \cap B$

3 Counting Cartesian Products

For two sets A and B, define the Cartesian Product as $A \times B = \{(a,b) : a \in A, b \in B\}$.

(a) Given two countable sets A and B, prove that $A \times B$ is countable.

(b) Given a finite number of countable sets $A_1, A_2, \ldots, A_n$, prove that $A_1 \times A_2 \times \cdots \times A_n$ is countable.

(c) Consider an infinite number of countable sets: $B_1, B_2, \ldots$. Under what condition(s) is $B_1 \times B_2 \times \cdots$ countable? Prove that if this condition is violated, $B_1 \times B_2 \times \cdots$ is uncountable.

4 Impossible Programs

Show that none of the following programs can exist.

(a) Consider a program P that takes in any program F, input x and output y and returns true if $F(x)$ outputs y and returns false otherwise.

(b) Consider a program P that takes in any program F and returns true if $F(F)$ halts and returns false if it doesn't halt.

(c) Consider a program P that takes in any programs F and G and returns true if F and G halt on all the same inputs and returns false otherwise.

5 Printing All x Where $M(x)$ Halts

Prove that it is possible to write a program P which:

- takes as input M, a Java program,
- runs forever, and prints out strings to the console,
- for every x, if $M(x)$ halts, then $P(M)$ eventually prints out x,
- for every x, if $M(x)$ does NOT halt, then $P(M)$ never prints out x.

6 Counting, Counting, and More Counting

The only way to learn counting is to practice, practice, practice, so here is your chance to do so. For this problem, you do not need to show work that justifies your answers. We encourage you to leave your answer as an expression (rather than trying to evaluate it to get a specific number).

(a) How many 10-bit strings are there that contain exactly 4 ones?

(b) How many ways are there to arrange n 1s and k 0s into a sequence?

(c) A bridge hand is obtained by selecting 13 cards from a standard 52-card deck. The order of the cards in a bridge hand is irrelevant.
How many different 13-card bridge hands are there? How many different 13-card bridge hands are there that contain no aces? How many different 13-card bridge hands are there that contain all four aces? How many different 13-card bridge hands are there that contain exactly 6 spades?

(d) How many 99-bit strings are there that contain more ones than zeros?

(e) An anagram of FLORIDA is any re-ordering of the letters of FLORIDA, i.e., any string made up of the letters F, L, O, R, I, D, and A, in any order. The anagram does not have to be an English word.
How many different anagrams of FLORIDA are there? How many different anagrams of ALASKA are there? How many different anagrams of ALABAMA are there? How many different anagrams of MONTANA are there?

(f) If we have a standard 52-card deck, how many ways are there to order these 52 cards?

(g) Two identical decks of 52 cards are mixed together, yielding a stack of 104 cards. How many different ways are there to order this stack of 104 cards?

(h) We have 9 balls, numbered 1 through 9, and 27 bins. How many different ways are there to distribute these 9 balls among the 27 bins? Assume the bins are distinguishable (e.g., numbered 1 through 27).

(i) We throw 9 identical balls into 7 bins. How many different ways are there to distribute these 9 balls among the 7 bins such that no bin is empty? Assume the bins are distinguishable (e.g., numbered 1 through 7).

(j) How many different ways are there to throw 9 identical balls into 27 bins? Assume the bins are distinguishable (e.g., numbered 1 through 27).

(k) There are exactly 20 students currently enrolled in a class. How many different ways are there to pair up the 20 students, so that each student is paired with one other student?

(l) Let (1, 1) be the bottom-left corner and (8, 8) be the upper-right corner of a chessboard. If you are allowed to move one square at a time and can only move up or right, what is the number of ways to go from the bottom-left corner to the upper-right corner?

(m) What is the number of ways to go from the bottom-left corner to the upper-right corner of the chesssboard, if you must pass through the square (6, 2), where (i, j) represents the square in the ith row from the bottom and the jth column from the left?

(n) How many solutions does $x_0 + x_1 + \cdots + x_k = n$ have, if each x must be a non-negative integer?

(o) How many solutions does $x_0 + x_1 = n$ have, if each x must be a *strictly positive* integer?

(p) How many solutions does $x_0 + x_1 + \cdots + x_k = n$ have, if each x must be a *strictly positive* integer?

7 Fermat's Necklace

Let p be a prime number and let k be a positive integer. We have an endless supply of beads. The beads come in k different colors. All beads of the same color are indistinguishable.

(a) We have a piece of string. As a relaxing study break, we want to make a pretty garland by threading p beads onto the string. How many different ways are there construct such a sequence of p beads of k different colors?

(b) Now let's add a restriction. We want our garland to be exciting and multicolored. Now how many different sequences exist? (Your answer should be a simple function of k and p.)

(c) Now we tie the two ends of the string together, forming a circular necklace which lets us freely rotate the beads around the necklace. We'll consider two necklaces equivalent if the sequence of colors on one can be obtained by rotating the beads on the other. (For instance, if we have $k = 3$ colors—red (R), green (G), and blue (B)—then the length $p = 5$ necklaces RGGBG, GGBGR, GBGRG, BGRGG, and GRGGB are all equivalent, because these are cyclic shifts of each other.)

How many non-equivalent sequences are there now? Again, the p beads must not all have the same color. (Your answer should be a simple function of k and p.)

[*Hint*: What follows if rotating all the beads on a necklace to another position produces an identical looking necklace?]

(d) Use your answer to part (c) to prove Fermat's little theorem. (Recall that Fermat's little theorem says that if p is prime and $a \not\equiv 0 \pmod{p}$, then $a^{p-1} \equiv 1 \pmod{p}$.)