

1 Sundry

Before you start your homework, write down your team. Who else did you work with on this homework? List names and email addresses. (In case of homework party, you can also just describe the group.) How did you work on this homework? Working in groups of 3-5 will earn credit for your "Sundry" grade.

Please copy the following statement and sign next to it:

I certify that all solutions are entirely in my words and that I have not looked at another student's solutions. I have credited all external sources in this write up.

2 Sum of Inverses

Prove that for every positive integer k , the following is true:

For every real number $r > 0$, there are only finitely many solutions in positive integers to

$$\frac{1}{n_1} + \cdots + \frac{1}{n_k} = r.$$

In other words, there exists some number m (that depends on k and r) such that there are at most m ways of choosing a positive integer n_1 , and a (possibly different) positive integer n_2 , etc., that satisfy the equation.

3 Stable Marriage

Consider a set of four men and four women with the following preferences:

men	preferences	women	preferences
A	1>2>3>4	1	D>A>B>C
B	1>3>2>4	2	A>B>C>D
C	1>3>2>4	3	A>B>C>D
D	3>1>2>4	4	A>B>D>C

- (a) Run on this instance the stable matching algorithm presented in class. Show each stage of the algorithm, and give the resulting matching, expressed as $\{(M, W), \dots\}$.
- (b) We know that there can be no more than n^2 stages of the algorithm, because at least one woman is deleted from at least one list at each stage. Can you construct an instance with n men and n women so that $c \cdot n^2$ stages are required for some respectably large constant c ? (We are looking for a *general pattern* here, one that results in $c \cdot n^2$ stages for any n .)
- (c) Suppose we relax the rules for the men, so that each unpaired man proposes to the next woman on his list at a time of his choice (some men might procrastinate for several days, while others might propose and get rejected several times in a single day). Can the order of the proposals change the resulting pairing? Give an example of such a change or prove that the pairing that results is the same.

4 Bieber Fever

In this world, there are only two kinds of people: people who love Justin Bieber, and people who hate him. We are searching for a stable matching for everyone. The situation is as follows:

- For some $n \geq 5$, there are n men, n women, and one Justin Bieber¹.
- Men can be matched with women; or anyone can be matched with Justin Bieber.
- Everyone is either a Hater or a Belieber. Haters want to be matched with anyone but Justin Bieber. Beliebers really want to be matched with Justin Bieber but don't mind being matched with other people.
- Men and women still have preference lists, as usual, but if they are a Belieber, Justin Bieber is always in the first position. If they are a hater, Justin Bieber is always in the last position.
- Justin Bieber desires to have 10 individuals matched with him (to party forever). As Justin Bieber is a kind person and wishes to be inclusive, he wishes to have exactly 5 women and 5 men in his elite club.
- Justin Bieber also has a preference list containing all $2n$ men and women.

A stable matching is defined as follows:

- Justin Bieber has 10 partners, of which 5 are men and 5 are women.

¹For the purposes of this problem, Justin Bieber is neither male nor female.

- All men and women not matched up with Justin Bieber are married to someone of the opposite gender.
- No rogue couples exist; i.e., there is no man M and woman W such that M prefers W to his current wife, and W prefers M to her current husband.
- No Hater is matched with Justin Bieber.
- There is no man who (1) is not matched with Justin Bieber; and (2) who is preferred by Justin Bieber over one of his current male partners; and (3) who prefers Justin Bieber over his wife. And similarly for women vis-a-vis Justin Bieber relative to their husbands and Justin Bieber's female partners.

- (a) Show that there does not necessarily exist a stable matching.
- (b) Provide an “if-and-only-if” condition for whether a stable matching exists. (*No need to prove anything in this part. That comes in later parts of this question.*)
- (c) Is Justin Bieber guaranteed to always get his Bieber-optimal group if a stable matching exists? (Bieber-optimal means that he gets the best possible group that could be matched to him in any stable matching.)
- (d) Give an algorithm which finds a stable matching if the condition you gave in (b) holds. Argue why this algorithm works.
- (e) Prove that a stable matching cannot exist if the condition you gave in (b) does not hold.

5 Combining Stable Marriages

In this problem we examine a simple way to *combine* two different solutions to a stable marriage problem. Let R, R' be two distinct stable matchings. Define the new matching $R \wedge R'$ as follows:

For every man m , m 's date in $R \wedge R'$ is whichever is better (according to m 's preference list) of his dates in R and R' .

Also, we will say that a man/woman *prefers* a matching R to a matching R' if he/she prefers his/her date in R to his/her date in R' . We will use the following example:

men	preferences	women	preferences
A	1>2>3>4	1	D>C>B>A
B	2>1>4>3	2	C>D>A>B
C	3>4>1>2	3	B>A>D>C
D	4>3>2>1	4	A>B>D>C

- (a) $R = \{(A, 4), (B, 3), (C, 1), (D, 2)\}$ and $R' = \{(A, 3), (B, 4), (C, 2), (D, 1)\}$ are stable matchings for the example given above. Calculate $R \wedge R'$ and show that it is also stable.

- (b) Prove that, for any matchings R, R' , no man prefers R or R' to $R \wedge R'$.
- (c) Prove that, for any stable matchings R, R' where m and w are dates in R but not in R' , one of the following holds:

- m prefers R to R' and w prefers R' to R ; or
- m prefers R' to R and w prefers R to R' .

[*Hint:* Let M and W denote the sets of men and women respectively that prefer R to R' , and M' and W' the sets of men and women that prefer R' to R . Note that $|M| + |M'| = |W| + |W'|$. (Why is this?) Show that $|M| \leq |W'|$ and that $|M'| \leq |W|$. Deduce that $|M'| = |W|$ and $|M| = |W'|$. The claim should now follow quite easily.]

(You may assume this result in subsequent parts even if you don't prove it here.)

- (d) Prove an interesting result: for any stable matchings R, R' , (i) $R \wedge R'$ is a matching [*Hint:* use the results from (c)], and (ii) it is also stable.

6 Better Off Alone

In the stable marriage problem, suppose that some men and women have standards and would not just settle for anyone. In other words, in addition to the preference orderings they have, they prefer being alone to being with some of the lower-ranked individuals (in their own preference list). A pairing could ultimately have to be partial, i.e., some individuals would remain single.

The notion of stability here should be adjusted a little bit. A pairing is stable if

- there is no paired individual who prefers being single over being with his/her current partner,
- there is no paired man and single woman (or paired woman and single man) that would both prefer to be with each other over being single or with his/her current partner,
- there is no paired man and paired woman that would both prefer to be with each other over their current partners, and
- there is no single man and single woman that would both prefer to be with each other over being single.

- (a) Prove that a stable pairing still exists in the case where we allow single individuals. You can approach this by introducing imaginary mates that people “marry” if they are single. How should you adjust the preference lists of people, including those of the newly introduced imaginary ones for this to work?
- (b) As you saw in the lecture, we may have different stable pairings. But interestingly, if a person remains single in one stable pairing, s/he must remain single in any other stable pairing as well (there really is no hope for some people!). Prove this fact by contradiction.

7 Quantitative Stable Marriage Algorithm

Once you have practiced the basic algorithm, let's quantify stable marriage problem a little bit. Here we define the following notation: on day j , let $P_j(M)$ be the rank of the woman that man M proposes to (where the first woman on his list has rank 1 and the last has rank n). Also, let $R_j(W)$ be the total number of men that woman W has rejected up through day $j - 1$ (i.e. not including the proposals on day j). Please answer the following questions using the notation above.

- (a) Prove or disprove the following claim: $\sum_M P_j(M) - \sum_W R_j(W)$ is independent of j . If it is true, please also give the value of $\sum_M P_j(M) - \sum_W R_j(W)$. The notation, \sum_M and \sum_W , simply means that we are summing over all men and all women.
- (b) Prove or disprove the following claim: one of the **men or women** must be matched to someone who is ranked in the top half of their preference list. You may assume that n is even.

8 Short Answer: Graphs

- (a) Bob removed a degree 3 node in an n -vertex tree, how many connected components are in the resulting graph? (An expression that may contain n .)
- (b) Given an n -vertex tree, Bob added 10 edges to it, then Alice removed 5 edges and the resulting graph has 3 connected components. How many edges must be removed to remove all cycles in the resulting graph? (An expression that may contain n .)
- (c) Give a gray code for 3-bit strings. (Recall, that a gray code is a sequence of the strings where adjacent elements differ by one. For example, the gray code of 2-bit strings is 00, 01, 11, 10. Note the last string is considered adjacent to the first and 10 differs in one bit from 00. Answer should be sequence of three-bit strings: 8 in all.)
- (d) For all $n \geq 3$, the complete graph on n vertices, K_n has more edges than the d -dimensional hypercube for $d = n$. (True or False.)
- (e) The complete graph with n vertices where n is an odd prime can have all its edges covered with x Rudrata cycles: a cycle where each vertex appears exactly once. What is the number, x , of such cycles in a cover? (Answer should be an expression that depends on n .)
- (f) Give a set of disjoint Rudrata cycles that covers the edges of K_5 , the complete graph on 5 vertices. (Each path should be a sequence (or list) of edges in K_5 .)