1 Party Tricks

You are at a party celebrating your completion of the CS 70 midterm. Show off your modular arithmetic skills and impress your friends by quickly figuring out the last digit(s) of each of the following numbers:

(a) Find the last digit of 11^{3142}.

(b) Find the last digit of 9^{9999}.

(c) Find the last digit of 3^{641}.

2 Modular Potpourri

(a) Evaluate $4^{96} \pmod 5$.

(b) Prove or Disprove: There exists some $x \in \mathbb{Z}$ such that $x \equiv 3 \pmod{16}$ and $x \equiv 4 \pmod 6$.

(c) Prove or Disprove: $2x \equiv 4 \mod 12 \iff x \equiv 2 \pmod{12}$.

3 Paper GCD

Given a sheet of paper such as this one, and no rulers, describe a method to find the GCD of the width and the height of the paper. You can fold or tear the paper however you want, and ultimately you should produce a square piece whose side lengths are equal to the GCD.

4 Extended Euclid

In this problem we will consider the extended Euclid's algorithm.

1. Calculate the gcd of 17 and 38, and determine how to express this as a "combination" of 17 and 38.
2. What does this imply, in this case, about the multiplicative inverse of 17, in arithmetic mod 38?
3. Note that x mod y, by definition, is always x minus a multiple of y. So, in the execution of Euclid's algorithm, each newly introduced value can always be expressed as a "combination"

of the previous two, like so:

$$\begin{aligned}
\gcd(2328,440) &= \gcd(440,128) & [\mathbf{128} = \mathbf{2328} - 5 \times \mathbf{440}] \\
&= \gcd(128,56) & [\mathbf{56} = \mathbf{440} - \underline{\quad\quad} \times \mathbf{128}] \\
&= \gcd(56,16) & [\mathbf{16} = \mathbf{128} - \underline{\quad\quad} \times \mathbf{56}] \\
&= \gcd(16,8) & [\mathbf{8} = \mathbf{56} - \underline{\quad\quad} \times \mathbf{16}] \\
&= \gcd(8,0) & [\mathbf{0} = \mathbf{16} - 2 \times \mathbf{8}] \\
&= 8.
\end{aligned}$$

(Fill in the blanks.)

4. Now working back up from the bottom, we will express the final gcd above as a combination of the two arguments on each of the previous lines:

$$\begin{aligned}
8 &= 1 \times \mathbf{8} + 0 \times \mathbf{0} = 1 \times \mathbf{8} + (\mathbf{16} - 2 \times \mathbf{8}) \\
&= 1 \times \mathbf{16} - 1 \times \mathbf{8} \\
&= \underline{\quad\quad} \times \mathbf{56} + \underline{\quad\quad} \times \mathbf{16}
\end{aligned}$$

[*Hint*: Remember, $\mathbf{8} = \mathbf{56} - 3 \times \mathbf{16}$. Substitute this into the above line.]

$$= \underline{\quad\quad} \times \mathbf{128} + \underline{\quad\quad} \times \mathbf{56}$$

[*Hint*: Remember, $\mathbf{16} = \mathbf{128} - 2 \times \mathbf{56}$.]

$$\begin{aligned}
&= \underline{\quad\quad} \times \mathbf{440} + \underline{\quad\quad} \times \mathbf{128} \\
&= \underline{\quad\quad} \times \mathbf{2328} + \underline{\quad\quad} \times \mathbf{440}
\end{aligned}$$