1 Key Facts

Key Facts are included to jog your memory. Your TA will explain these in detail.

We have covered several types of proofs so far, the most recent of which was *induction*.

1. **Direct Proof**: Prove a statement using a collection of other facts you know.
2. **Contraposition**: Consider $P \implies Q$. Then, prove $\neg Q \implies \neg P$.
3. **Contradiction**: Consider P. Assume otherwise and show it leads to a contradiction.
4. **Induction**: Use a base case, inductive hypothesis, and inductive step.

What is **strong induction**? As it turns out, strong induction is simply changing our inductive hypothesis: assume the *first k* are true, instead of assuming only the *previous* $(k-1)$*th* is true.

How do know when to **strengthen our inductive hypothesis**? In general, strengthen when you're stuck. (e.g., instead of proving $a_{i+1} = a_i^2 - a_i + 1 \geq 0$, where $a_0 = 1$, prove $0 \leq a_i^2 - 2a_i + 1$. Hint: How to simplify $a_i^2 - 2a_i + 1$?)

2 Fibonacci Proof

Let F_i be the i^{th} Fibonacci number, defined by $F_{i+2} = F_{i+1} + F_i$ and $F_0 = 0$, $F_1 = 1$. Prove that

$$\sum_{i=0}^{n} F_i^2 = F_n F_{n+1}.$$

3 Induction

Prove the following using induction:

(a) For all natural numbers $n > 2$, $2^n > 2n + 1$.

(b) For all positive integers n, $1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 = n^2(2n^2 - 1)$.

(c) For all natural numbers n, $(5/4)8^n + 3^{3n-1}$ is divisible by 19.

4 Seating Arrangement

N people have come to watch a play and were given a row with exactly N consecutive seats. They have decided on the following seating arrangement. After the first person sits down, the next person has to sit next to the first. The third sits next to one of the first two and so on until all N are seated. In other words, no person can take a seat that separates him/her from at least one other person. How many different ways can this be accomplished? Note that the first person can choose any of the N seats. (Hint: Use induction.)

5 Make It Stronger

Suppose that the sequence $a_1, a_2, \ldots$ is defined by $a_1 = 1$ and $a_{n+1} = 3a_n^2$ for $n \geq 1$. We want to prove that

$$a_n \leq 3^{2^n}$$

for every natural number n.

1. Suppose that we want to prove this statement using induction, can we let our induction hypothesis be simply $a_n \leq 3^{2^n}$? Show why this does not work.
2. Try to instead prove the statement $a_n \leq 3^{2^n - 1}$ using induction. Does this statement imply what you tried to prove in the previous part?

6 Bit String

Prove that every positive integer n can be written with a string of 0s and 1s. In other words, prove that we can write

$$n = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \ldots + c_1 \cdot 2^1 + c_0 \cdot 2^0,$$

where $k \in \mathbb{N}$ and $c_k \in \{0, 1\}$.